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Stability of Best Rational Chebyshev Approximation

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In this paper we consider best Chebyshev approximation to continuous functions by generalized rational functions using an optimization theoretical approach introduced in [B. Brosowski and C. Guerreiro, On the characterization of a set of optimal points and some applications, *in* "Approximation and Optimization in Mathematical Physics" (B. Brosowski and E. Martensen, Eds.), pp. 141–174, Verlag Peter Lang, Frankfurt a.M./Bern, 1983]. This general approach includes, in a unified way, usual, weighted, one-sided, unsymmetric, and also more general rational Chebychev approximation problems with side-conditions. We derive various continuity conditions for the optimal value, for the feasible set, and the optimal set of the corresponding optimization problem. From these results we derive conditions for the upper semicontinuity of the metric projection, which include some of the results of Werner [On the rational Tschebyscheff operator, *Math. Z.* **86** (1964), 317–326] and Cheney and Loeb [On the continuity of rational approximation operators, *Arch. Rational Mech. Anal.* **21** (1966), 391–401].

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1. Introduction

Let S be a compact Hausdorff space, $S \neq \emptyset$, and consider the compact Hausdorff space $T := \{-1, 1\} \times S$. Let $\sigma := (B, C, \gamma, x)$ where

(i) $B, C: S \to \mathbb{R}^N$ are continuous functions such that the convex and open set

$$U_\sigma := U_C := \bigcap_{s \in S} \{v \in \mathbb{R}^N \mid \langle C(s), v \rangle > 0\}$$

is non-empty, $N \in \mathbb{N}$,

(ii) $\gamma: T \to \mathbb{R}$ is a non-negative continuous function such that

$$\forall_{s \in S} \; \gamma(-1, s) + \gamma(1, s) > 0,$$

(iii) $x: S \to \mathbb{R}$ is a continuous function.

We denote by $\mathfrak{P}$ the set of all such "parameters" σ and define for each $\sigma \in \mathfrak{P}$ a norm by setting

$$\|\sigma\| := \max\{\|B\|_\infty, \|C\|_\infty, \|\gamma\|_\infty, \|x\|_\infty\}.$$

For real valued functions on S or T, $\|\cdot\|_\infty$ denotes the usual sup norm and, for vector valued functions $A: S \to \mathbb{R}^N$, $\|\cdot\|_\infty$ is defined by

$$\|A\|_\infty := \sup\{\|A(s)\| \in \mathbb{R} \mid s \in S\},$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^N$.
We denote by π_i the projection of a Cartesian product on its ith factor.

For each $(y, z) \in \mathbb{R}^N \times \mathbb{R}$ define $p(y, z) := z$ and consider the minimization problem MPR(σ)

$$\text{Minimize } p(v, z) \text{ subject to } v \in U_\sigma \text{ and}$$

$$\forall_{(\eta, s) \in T} \; \eta \frac{\langle B(s), v \rangle}{\langle C(s), v \rangle} - \gamma(\eta, s)\, z \leqslant \eta x(s).$$

As it was shown in [5], this minimization problem extends the classical rational Chebyshev approximation problem, compare also Example 1.1. It includes also weighted, one-sided, and unsymmetric approximation problems. Further we have shown in [5], that this minimization problem is non-quasi-convex and permits not only a local theory but also a global theory, compare [6, 7, 8].

Due to the nature of this problem, it suffices to consider in U_σ elements of norm 1. We will again denote by U_σ the set

$$U_\sigma := \{v \in \mathbb{R}^N \mid \|v\| = 1\}.$$

Further we define the set

$$V_\sigma := \left\{ r \in C(S) \,\middle|\, \underset{v \in U_\sigma}{\exists} \; r = \frac{\langle B, v \rangle}{\langle C, v \rangle} \right\}$$

and the continuous mapping $R_\sigma : U_\sigma \to V_\sigma$ by setting

$$R_\sigma(v) := \frac{\langle B, v \rangle}{\langle C, v \rangle}$$

for each $v \in U_\sigma$.

For each $\sigma \in \mathfrak{P}$ we define the sets

$$Z_\sigma := \bigcap_{(\eta, s) \in T} \left\{ (v, z) \in U_\sigma \times \mathbb{R} \,\middle|\, \eta \frac{\langle B(s), v \rangle}{\langle C(s), v \rangle} - \gamma(\eta, s)\, z \leqslant \eta x(s) \right\}$$

and

$$F_\sigma := \left\{ (r, z) \in V_\sigma \times \mathbb{R} \,\middle|\, \underset{(v, z) \in Z_\sigma}{\exists} \; r = \frac{\langle B, v \rangle}{\langle C, v \rangle} \right\},$$

which are called *feasible sets*. Further we introduce the *minimum value*

$$E_\sigma := \inf\{z \in \mathbb{R} \mid (v, z) \in Z_\sigma\}.$$

Since γ is not identically zero, we have $E_\sigma \geqslant 0$ provided $Z_\sigma \neq \varnothing$. The set of all solutions of MPR(σ) in Z_σ resp. F_σ is denoted by

$$P_\sigma := \{(v, z) \in Z_\sigma \mid z = E_\sigma\}$$

resp.

$$Q_\sigma := \{(r, z) \in F_\sigma \mid z = E_\sigma\}.$$

Further, we introduce

$$\mathfrak{M} := \{\sigma \in \mathfrak{P} \mid Z_\sigma \neq \varnothing\}$$

and the *solvability set*

$$\mathfrak{L} := \{\sigma \in \mathfrak{P} \mid P_\sigma \neq \varnothing\}.$$

Clearly, we have $\mathfrak{L} \subset \mathfrak{M}$. If $N=1$, then $\mathfrak{L}=\mathfrak{M}$. In fact, if a sequence (z_n) in $\mathbb{R}^+$ converges to E_σ and satisfies the inequalities

$$\underset{(\eta, s) \in T}{\forall} \eta \frac{B(s)}{C(s)} - \gamma(\eta, s)\, z_n \leqslant \eta\, x(s),$$

then we have for $n \to \infty$

$$\underset{(\eta, s) \in T}{\forall} \eta \frac{B(s)}{C(s)} - \gamma(\eta, s)\, E_\sigma \leqslant \eta\, x(s),$$

which implies that either $(1, E_\sigma)$ or $(-1, E_\sigma)$ in Z_σ.

We say that σ satisfies the *Slater condition*, if the set

$$Z_\sigma^< := \bigcap_{(\eta, s) \in T} \left\{ (v, z) \in Z_\sigma \,\middle|\, \eta \frac{\langle B(s), v\rangle}{\langle C(s), v\rangle} - \gamma(\eta, s)\, z < \eta x(s) \right\}$$

is non-empty, which is equivalent to the set

$$F_\sigma^< := \bigcap_{(\eta, s) \in T} \{ (r, z) \in F_\sigma \mid \eta r(s) - \gamma(\eta, s)\, z < \eta x(s) \}$$

being non-empty.

Let σ in $\mathfrak{M}$ be given. For each v_0 in U_σ define the linear space

$$H_{v_0} := H_{r_0} := \{ y \in \mathbb{R}^N \mid \langle r_0(s)\, C(s) - B(s), y\rangle = 0 \},$$

where $r_0 := \langle B, v_0\rangle / \langle C, v_0\rangle$.

An element v_0 in U_σ is called *normal* (with respect to σ) iff $\dim H_{v_0} = 1$. Let v_0 be normal. Then we also call

$$r_0 = \frac{\langle B, v_0\rangle}{\langle C, v_0\rangle} \qquad \text{and} \qquad (v_0, z_0) \in Z_\sigma$$

normal. A parameter $\sigma \in \mathfrak{L}$ is called normal, if every point in P_σ is normal (with respect to σ).

A particular case of MPR(σ) is given by the following example:

EXAMPLE 1.1. Let $g_0, g_1, ..., g_l, h_0, h_1, ..., h_m \in C[a, b]$ be such that the set

$$\left\{ \beta \in \mathbb{R}^{m+1} \,\middle|\, \underset{s \in [a, b]}{\forall} \sum_{i=0}^{m} \beta_i h_i(s) > 0 \right\}$$

is non-empty and define $N := l+m+2$,

$$B_0(s) := (g_0(s), g_1(s), ..., g_l(s), 0, 0, ..., 0),$$
$$C_0(s) := (0, 0, ..., 0, h_0(s), h_1(s), ..., h_m(s)),$$
$$\gamma_0(\eta, s) := 1.$$

For each function $x \in C[a, b]$, define the parameter $\sigma_x := (B_0, C_0, \gamma_0, x)$ and the set

$$\tilde{\mathfrak{L}}_0 := \{\sigma_x \in \mathfrak{L} \mid x \in C[a, b]\}.$$

Let (r, E_{σ_x}) in F_{σ_x} be a solution of $\mathrm{MPR}(\sigma_x)$. Then r is a Chebyshev approximation to the function x from the set of generalized rational functions

$$V := \left\{\frac{\langle B_0, v\rangle}{\langle C_0, v\rangle} \in C[a, b] \;\middle|\; \underset{s \in [a, b]}{\forall} \langle C_0(s), v\rangle > 0\right\}$$

with minimum distance E_{σ_x}. In this case we have

$$V = \pi_1(F_{\sigma_x}) = V_{\sigma_x},$$

i.e., the set of approximating functions is independent of the function x.

If we choose

$$\gamma_1(\eta, s) = \frac{1-\eta}{2},$$

we obtain the one-sided approximation problem. In this case the set of approximating functions is given by

$$\left\{\frac{\langle B_0, v\rangle}{\langle C_0, v\rangle} \in C[a, b] \;\middle|\; \underset{s \in [a, b]}{\forall} \langle C_0(s), v\rangle > 0 \text{ and } \frac{\langle B_0(s), v\rangle}{\langle C_0(s), v\rangle} \leqslant x(s)\right\}.$$

It is clear that in this case the set of approximating elements depends on the function $x \in C[a, b]$.

If we choose $g_\nu(s) := s^\nu$, $\nu = 0, 1, ..., l$ and $h_\nu(s) := s^\nu$, $\nu = 0, 1, ..., m$, we obtain the classical rational Chebyshev approximation problem.

In this paper we investigate the stability of the minimization problem $\mathrm{MPR}(\sigma)$, i.e., we investigate the continuity of the feasible set-mappings

$$Z: \mathfrak{M} \to \mathrm{POW}(S^{N-1} \times \mathbb{R}) \qquad \text{and} \qquad F: \mathfrak{M} \to \mathrm{POW}(C(S) \times \mathbb{R}),$$

the minimal set mappings

$$P: \mathfrak{L} \to \mathrm{POW}(S^{N-1} \times \mathbb{R}) \qquad \text{and} \qquad Q: \mathfrak{L} \to \mathrm{POW}(C(S) \times \mathbb{R}),$$

and of the minimal value

$$E: \mathfrak{L} \to \mathbb{R},$$

where POW(M) denotes the power set of a set M and S^{N-1} denotes the unit sphere in $\mathbb{R}^N$.

We will use the usual concepts of lower and upper semicontinuity for the set valued mappings:

DEFINITION 1.2. Let X, Y be metric spaces and $F: X \to \mathrm{POW}(Y)$ be a set valued mapping.

(i) The mapping F is lower semicontinuous at the point $x_0 \in X$ iff for each open set $W \subset Y$ such that $W \cap F(x_0) \neq \varnothing$, there exists an open set $W_0 \subset X$ such that $x_0 \in W_0$ and

$$x \in W_0 \Rightarrow F(x) \cap W \neq \varnothing.$$

(ii) The mapping F is upper semicontinuous at the point $x_0 \in X$ iff for each open set $W \subset Y$ such that $F(x_0) \subset W$, there exists an open set $W_0 \subset X$ such that $x_0 \in W_0$ and

$$x \in W_0 \Rightarrow F(x) \subset W.$$

Our investigations showed that due to the side condition $v \in U_\sigma$ the usual concept of a closed set-valued mapping is not so suitable for the investigation of the mappings Z and P. Thus, we used the following more suitable modification:

DEFINITION 1.3. Let $\mathfrak{N}$ be a non-empty subset of $\mathfrak{M}$. A set-valued mapping

$$\psi: \mathfrak{N} \to \mathrm{POW}(S^{N-1} \times \mathbb{R})$$

is called r-closed in $\sigma_0 \in \mathfrak{N}$ iff given sequences (σ_n) in $\mathfrak{N}$ and $(v_n, z_n) \in S^{N-1} \times \mathbb{R}$ such that

$$\sigma_n \to \sigma_0 \text{ and } (v_n, z_n) \to (v_0, z_0) \text{ and } \forall_{n \in \mathbb{N}} \; (v_n, z_n) \in \psi(\sigma_n) \text{ and } v_0 \in \psi(U_{\sigma_0}),$$

then $(v_0, z_0) \in \psi(\sigma_0)$.

For the classical rational Chebyshev approximation problem (compare Example 1.1) H. Maehly and Ch. Witzgall [14] considered the parameter set $\tilde{\mathfrak{L}}_0$ and proved that the metric projection

$$\pi_1 \circ Q: \tilde{\mathfrak{L}}_0 \to C[a, b]$$

is continuous at all normal points of $\tilde{\mathfrak{L}}_0$ and can be discontinuous at non-normal points. In this case $\pi_1 \circ Q$ is a point-to-point mapping since for all $\sigma_x \in \tilde{\mathfrak{L}}_0$ the problem has a unique solution in F_{σ_x}. This result was extended by H. Werner [17], who showed that at all non-normal points σ_x the metric projection $\pi_1 \circ Q$ is always discontinuous provided x is not contained in $\pi_1 \circ F_\sigma$. In this case $\pi_1 \circ Q$ is also continuous. Later E. W. Cheney and H. L. Loeb [12] considered Chebyshev approximation by generalized rational functions in the interval $[a, b]$ and they proved: If for each $\sigma_x \in \tilde{\mathfrak{L}}_0$ the problem MPR(σ_x) has a unique solution in F_{σ_x}, then the metric projection

$$\pi_1 \circ Q : \tilde{\mathfrak{L}}_0 \to C[a, b]$$

is continuous at σ_x if and only if $x \in \pi_1 \circ F_{\sigma_x}$ or x is normal. Later H. L. Loeb and D. G. Moursund [13] extended some of these results to restricted range approximation, which for a fixed parameter includes also one-sided best approximation. In this last case they defined for $x \in C(S)$ the set of approximating elements by

$$\left\{ r \in V_\sigma \,\middle|\, \underset{s \in S}{\forall}\ r(s) \leqslant x(s) \right\}$$

and considered best approximations to functions $y \in C(S)$ from this set. Thus, in their stability investigations they considered only variations of the function y, where the set of approximating elements is fixed. The same can be said for the linear case as the results of G. D. Taylor [16] and L. L. Schumaker and G. D. Taylor [15] show, compare also the review paper of B. L. Chalmers and G. D. Taylor [11]. For best approximation in normed linear spaces B. Brosowski, Deutsch, and Nürnberger [4] considered also variable subspaces and obtained some stability results.

In our investigation of the stability of the problem MPR(σ) we consider variations of all the coordinates of the parameter σ. Thus, we include also the case of a variable set of approximating functions. An important rôle is played by Slater condition, which is considered in detail in Section 2.

In Section 3 we show that the lower semicontinuity of Z and of F at a point $\sigma \in \mathfrak{M}$ are equivalent to the Slater condition in σ as well as to the upper semicontinuity of E at σ. It should be remarked that the proof of the implication

$$\sigma \text{ satisfies Slater condition} \Rightarrow Z(\text{or } F) \text{ is lower semicontinuous at } \sigma$$

is a slight extension of the classical proof for strictly quasiconvex minimization problems, compare Bank, Guddat, Klatte, and Tammer [1, pp. 40–41]. The implication

$$Z \text{ (or } F) \text{ lower semicontinuous at } \sigma \Rightarrow E \text{ upper semicontinuous at } \sigma$$

is also true for non-quasi-convex minimization problems, compare [1, pp. 60–62], and the references mentioned there. The proofs given here use the special structure of the minimization problem MPR(σ). It is remarkable that in the case of the non-quasi-convex problem MPR(σ), the converse implications are true. We have upper semicontinuity at a point $\sigma \in \mathfrak{M}$ for the mapping Z only in the case $N=1$. For $N \geqslant 2$ the mapping Z is never upper semicontinuous at a point σ in $\mathfrak{M}$. For all $\sigma \in \mathfrak{M}$ such that the set V_σ is nowhere dense in $C(S)$, the mapping F is not upper semicontinuous at σ.

It is well-known and easy to prove that in the case of ordinary Chebyshev approximation the minimal value is continuous (in fact it is Lipschitz continuous), if one considers only variations of the function x. This can also be derived from a more general result for MPR(σ), compare [9]. If one considers variation of all coordinates of σ, then the situation is much more difficult.

In Section 4 we prove, for the case $N=1$, that the continuity of E at $\sigma \in \mathfrak{L}$ is equivalent to Slater condition in σ. For the case $N \geqslant 2$, we prove

$$P \text{ upper semicontinuous at } \sigma \Rightarrow E \text{ continuous at } \sigma$$
$$\Rightarrow \sigma \text{ satisfies Slater condition}$$

and

$$P_\sigma \text{ compact and } \sigma \text{ satisfies Slater condition} \Rightarrow E \text{ continuous at } \sigma.$$

In Section 5 we consider the stability of the mapping P. Our main results are:

(i) The set

$$\tilde{\mathfrak{L}} := \{\sigma \in \mathfrak{L} \mid P_\sigma \text{ compact and } \sigma \text{ satisfies Slater condition}\}$$

is open in $\mathfrak{L}$.

(ii) P upper semicontinuous at $\sigma \Leftrightarrow \sigma \in \tilde{\mathfrak{L}}$.

For the proof of the necessity that $\sigma \in \tilde{\mathfrak{L}}$, we had to assume that $\# S \geqslant N-1$, i.e., the space S must contain enough points. Since P_σ compact implies σ normal (compare Proposition 5.5), the statement (ii) is similar to the results of H. Werner [17] and E. W. Cheney and H. L. Loeb [12] for the metric projection $\pi_1 \circ Q$ in the case of ordinary rational Chebyshev approximation in the interval $[a, b]$. We can derive from our statement one direction of their result, namely:

(iii) σ normal and $\# \pi_1 \circ Q_\sigma = 1 \Rightarrow \pi_1 \circ Q$ continuous at σ.

Even in this particular case our result is more general, since we permit variations of all coordinates of σ and do not assume $S=[a, b]$ and Haar condition, compare Corollary 6.7. The statement (iii) is a consequence of the main results of Section 6, where we consider the stability of the mapping Q. These main results are:

(iv) Q upper semicontinuous at $\sigma \Rightarrow Q_\sigma$ compact and σ satisfies Slater condition.

(v) Q compact and σ satisfies Slater condition and σ normal $\Rightarrow Q$ upper semicontinuous at σ,

(vi) The set

$$\{\sigma \in \mathfrak{L} \mid Q_\sigma \text{ compact and } \sigma \text{ satisfies Slater condition and } \sigma \text{ normal}\}$$

is open in $\mathfrak{L}$.

It is an open question whether the upper semicontinuity of Q at a point $\sigma \in \mathfrak{L}$ implies also the normality of σ as in the mentioned case of ordinary Chebyshev approximation in the interval $[a, b]$.

We excluded an investigation of the lower semicontinuity of P and Q, since according to the known results for linear case, compare B. Brosowski [2], this problem needs its own investigation.

2. On Slater's Condition

PROPOSITION 2.1. *If $\sigma \in \mathfrak{M}$ satisfies the Slater condition, then we have*

$$\overline{Z_\sigma^<} = Z_\sigma \qquad \textit{and} \qquad \overline{F_\sigma^<} = F_\sigma.$$

For the proof, see in [10, proof of Theorem 1.1].

Define for $(v, z) \in Z_\sigma$ the set

$$M(\sigma, v, z) := \left\{(\eta, s) \in T \,\middle|\, \eta \frac{\langle B(s), v\rangle}{\langle C(s), v\rangle} - \gamma(\eta, s)\, z = \eta x(s)\right\}$$

Using [8, Theorem 1.1; 10, Theorem 1.1], we have:

PROPOSITION 2.2. *Let $\sigma_0 := (B_0, C_0, \gamma_0, x_0)$ and $\sigma = (B_0, C_0, \gamma, x)$ be such that σ satisfies the Slater condition.*

If $(v_0, E_{\sigma 0}) \in P_{\sigma 0}$, $(v_0, z) \in Z_\sigma$, and $M(\sigma_0, v_0, E_{\sigma 0}) \subset M(\sigma, v_0, z)$, then $(v_0, z) \in P_\sigma$.

Remark. The theorems used from [8, 10] assume $x_0 \notin V_{\sigma 0}$. If $x_0 \in V_{\sigma 0}$, then the result is trivial.

PROPOSITION 2.3. *If* $(v_0, z_0) \in Z_{\sigma_0}$, *resp.* $(r_0, z_0) \in F_{\sigma_0}$, *is a Slater element for* $\sigma_0 \in \mathfrak{M}$, *then there exists a neighborhood* W *of* σ_0 *in* $\mathfrak{M}$ *such that*

$$\forall_{\sigma \in W} \ (v_0, z_0) \in Z_\sigma^<, \qquad \text{resp.} \qquad (r_0, z_0) \in F_\sigma^<.$$

Proof. If we set $r_0 := \langle B_0, v_0 \rangle / \langle C_0, v_0 \rangle$, the following proof works in both cases. Since

$$\forall_{(\eta, s) \in T} \ \eta \frac{\langle B_0(s), v_0 \rangle}{\langle C_0(s), v_0 \rangle} - \gamma_0(\eta, s)\, z_0 < \eta x_0(s),$$

there exists a real number $\delta > 0$ such that

$$\forall_{(\eta, s) \in T} \ \eta \frac{\langle B_0(s), v_0 \rangle}{\langle C_0(s), v_0 \rangle} - \gamma_0(\eta, s)\, z_0 - \eta x_0(s) \leqslant -\delta < 0,$$

we can also assume that

$$\forall_{s \in S} \ \langle C_0(s), v_0 \rangle \geqslant \delta.$$

Define

$$\varepsilon := \min\left\{\frac{\delta}{4}, \frac{\delta}{4(z_0 + 1)}, \frac{\delta^3}{8} \cdot (\| B_0 \|_\infty + \| C_0 \|_\infty)^{-1}\right\}$$

and

$$W := \{\sigma \in \mathfrak{M} \mid \|\sigma - \sigma_0\| < \varepsilon\}.$$

If $\sigma \in W$ and $(\eta, s) \in T$ we have

$$\begin{aligned} \langle C(s), v_0 \rangle &= \langle C_0(s), v_0 \rangle + \langle C(s) - C_0(s), v_0 \rangle \\ &\geqslant \delta - \| C - C_0 \|_\infty \geqslant \delta/2 \end{aligned}$$

and

$$\begin{aligned} &\eta \frac{\langle B(s), v_0 \rangle}{\langle C(s), v_0 \rangle} - \gamma(\eta, s)\, z_0 - \eta x(s) \\ &\quad = \eta \left[\frac{\langle B(s), v_0 \rangle}{\langle C(s), v_0 \rangle} - \frac{\langle B_0(s), v_0 \rangle}{\langle C_0(s), v_0 \rangle}\right] \\ &\qquad + \left[\eta \frac{\langle B_0(s), v_0 \rangle}{\langle C_0(s), v_0 \rangle} - \gamma_0(\eta, s)\, z_0 - \eta x_0(s)\right] \\ &\qquad + [\gamma_0(\eta, s) - \gamma(\eta, s)]\, z_0 + \eta[x_0(s) - x(s)] \end{aligned}$$

$$\leqslant \frac{\langle C_0(s), v_0\rangle \cdot |\langle B(s)-B_0(s), v_0\rangle| + |\langle B_0(s), v_0\rangle|\,|\langle C(s)-C_0(s)\, v_0\rangle|}{\langle C(s), v_0\rangle \cdot \langle C_0(s), v_0\rangle}$$

$$-\delta + \|\gamma_0-\gamma\|_\infty z_0 + \|x_0-x\|_\infty$$

$$\leqslant \frac{2}{\delta^2}(\|C_0\|_\infty + \|B_0\|_\infty)\cdot\varepsilon - \delta + \varepsilon z_0 + \varepsilon$$

$$\leqslant \frac{\delta}{4} - \delta + \frac{\delta}{4} + \frac{\delta}{4} = -\frac{\delta}{4} < 0. \quad \blacksquare$$

PROPOSITION 2.4. *Let Z^* be a non-empty subset of Z_σ such that*

$$(v_1, z_1), (v_2, z_2) \in Z^* \Rightarrow \underset{\rho\in[0,1]}{\forall}\ \underset{z\geqslant\max(z_1,z_2)}{\forall} \left(\frac{\rho v_1 + (1-\rho) v_2}{\|\rho v_1 + (1-\rho) v_2\|}, z\right) \in Z^*$$

and the set

$$V^* := \left\{ v \in \mathbb{R}^N \,\middle|\, \underset{z\in\mathbb{R}}{\exists} \left(\frac{v}{\|v\|}, z\right) \in Z^* \right\}$$

is convex.

Assume Z^* does not satisfy the Slater condition, i.e., there does not exist an element $(v. z)$ in Z^* such that

$$\underset{(\eta,s)\in T}{\forall}\ \eta\frac{\langle B(s), v\rangle}{\langle C(s), v\rangle} - \gamma(\eta, s)\, z < \eta x(s).$$

Then:

(i) *The set*

$$T^* := \bigcap_{(v,z)\in Z^*} \left\{ (\eta, s) \in T \,\middle|\, \frac{\langle B(s), v\rangle}{\langle C(s), v\rangle} = x(s) \text{ and } \gamma(\eta, s) = 0 \right\}$$

is non-empty and

(ii) $$\underset{(v,z)\in U_\sigma\times\mathbb{R}}{\forall}\ \underset{(\eta,s)\in T^*}{\exists}\ \eta\frac{\langle B(s), v\rangle}{\langle C(s), v\rangle} \geqslant \eta x(s).$$

Proof. (i) Choose an element (v_0, z_0) in $V^* \times \mathbb{R}$ such that

$$v_0 \in \text{relint}\,(V^*) \text{ and } z_0 > \inf\left\{ z \in \mathbb{R} \,\middle|\, \left(\frac{v_0}{\|v_0\|}, z\right) \in Z^* \right\}.$$

Since the Slater-condition is not fulfilled, there is an element (η, s) in T such that

$$\eta\frac{\langle B(s), v_0\rangle}{\langle C(s), v_0\rangle} - \gamma(\eta, s)\, z_0 = \eta x(s). \qquad (*)$$

There exists an $\varepsilon > 0$ such that the element $(v_0, z_0 - \varepsilon)$ is also feasible, i.e., we have

$$\eta \frac{\langle B(s), v_0 \rangle}{\langle C(s), v_0 \rangle} - \gamma(\eta, s)(z_0 - \varepsilon) \leqslant \eta x(s).$$

Subtracting $(*)$ from the last inequality we obtain

$$\gamma(\eta, s)\varepsilon \leqslant 0,$$

which implies

$$\gamma(\eta, s) = 0 \qquad \text{and} \qquad \frac{\langle B(s), v_0 \rangle}{\langle C(s), v_0 \rangle} = x(s).$$

Choose any element $v \in V^*$. Since $v_0 \in \operatorname{relint}(V^*)$, there exist an element $v_1 \in V^*$ and a real number $0 < \rho < 1$ such that

$$v_0 = \rho v + (1 - \rho) v_1.$$

The element v and v_1 satisfy the inequalities

$$\eta \frac{\langle B(s), v \rangle}{\langle C(s), v \rangle} - \eta x(s) \leqslant 0$$

and

$$\eta \frac{\langle B(s), v_1 \rangle}{\langle C(s), v_1 \rangle} - \eta x(s) \leqslant 0.$$

Then we have

$$\begin{aligned} 0 &= \eta \frac{\langle B(s), v_0 \rangle}{\langle C(s), v_0 \rangle} - \eta x(s) \\ &= \rho \frac{\langle C(s), v \rangle}{\langle C(s), v_0 \rangle} \left[\eta \frac{\langle B(s), v \rangle}{\langle C(s), v \rangle} - \eta x(s) \right] \\ &\quad + (1 - \rho) \frac{\langle C(s), v_1 \rangle}{\langle C(s), v_0 \rangle} \left[\eta \frac{\langle B(s), v_1 \rangle}{\langle C(s), v_1 \rangle} - \eta x(s) \right] \\ &\leqslant \rho \frac{\langle C(s), v \rangle}{\langle C(s), v_0 \rangle} \left[\eta \frac{\langle B(s), v \rangle}{\langle C(s), v \rangle} - \eta x(s) \right] \leqslant 0. \end{aligned}$$

Thus, it follows $\langle B(s), v \rangle / \langle C(s), v \rangle = x(s)$.

Since v was chosen arbitrarily in V^*, the set

$$\bigcap_{v \in V^*} \left\{ (\eta, s) \in T \,\middle|\, \frac{\langle B(s), v\rangle}{\langle C(s), v\rangle} = x(s) \text{ and } \gamma(\eta, s) = 0 \right\}$$

is non-empty and clearly equal to T^*.

(ii) Assume there exists an element $(v_1, z_1) \in U_\sigma \times \mathbb{R}$ such that

$$\forall_{(\eta, s) \in T^*} \eta \frac{\langle B(s), v_1\rangle}{\langle C(s), v_1\rangle} < \eta x(s).$$

By compactness of T^* and continuity of the functions involved, there exists an open set $W^* \supset T^*$ such that

$$\forall_{(\eta, s) \in W^*} \eta \frac{\langle B(s), v_1\rangle}{\langle C(s), v_1\rangle} - \gamma(\eta, s)\, z_1 < \eta x(s).$$

Since $T^* \neq T$, we can assume that W^* is different from T.

We claim that there exists an element (v_2, z_2) in Z^* such that

$$\forall_{(\eta, s) \in T \setminus W^*} \eta \frac{\langle B(s), v_2\rangle}{\langle C(s), v_2\rangle} - \gamma(\eta, s)\, z_2 < \eta x(s).$$

If not, consider the set

$$\tilde{Z} := \bigcap_{(\eta, s) \in T \setminus W^*} \left\{ (v, z) \in S^{N-1} \times \mathbb{R} \,\middle|\, \eta \frac{\langle B(s), v\rangle}{\langle C(s), v\rangle} - \gamma(\eta, s)\, z \leqslant \eta x(s) \text{ and } \langle C(s), v\rangle > 0 \right\}.$$

Since $Z^* \subset \tilde{Z}$, applying part (i), the set

$$\tilde{T} := \bigcap_{(v, z) \in Z^*} \left\{ (\eta, s) \in T \setminus W^* \,\middle|\, \frac{\langle B(s), v\rangle}{\langle C(s), v\rangle} = x(s) \text{ and } \gamma(n, s) = 0 \right\}$$

is non-empty. By definition of T^*, we have $\tilde{T} \subset T^*$, which is not possible. Thus, the claim is proved.

By compactness of $T \setminus W^*$ there exist M, $K > 0$ such that

$$\forall_{(\eta, s) \in T \setminus W^*} \eta \frac{\langle B(s), v_1\rangle}{\langle C(s), v_1\rangle} - \gamma(\eta, s)\, z_1 \leqslant \eta x(s) + K$$

$$\text{and } \eta \frac{\langle B(s), v_2\rangle}{\langle C(s), v_2\rangle} - \gamma(\eta, s)\, z_2 \leqslant \eta x(s) - M.$$

We remark that all the above inequalities remain true if we replace z_1, z_2 by

$$z := \max\{z_1, z_2\}.$$

Choose

$$0 < \rho < \frac{M}{M + K\alpha} < 1,$$

where

$$\alpha := \max\left\{\frac{\langle C(s), v_1\rangle}{\langle C(s), v_2\rangle} \in \mathbb{R} \,\middle|\, s \in S\right\},$$

and define $v := \rho v_1 + (1-\rho) v_2$. We will show that $(v/\|v\|, z)$ is a Slater element for Z^*, which is a contradiction.

For each $(\eta, s) \in W^*$ we have

$$\eta \frac{\langle B(s), v_1\rangle}{\langle C(s), v_1\rangle} - \gamma(\eta, s)\, z < \eta x(s)$$

and

$$\eta \frac{\langle B(s), v_2\rangle}{\langle C(s), v_2\rangle} - \gamma(\eta, s)\, z \leqslant \eta x(s).$$

Multiplying the first inequality by $\rho\langle C(s), v_1\rangle/\langle C(s), v\rangle$, the second by $(1-\rho)\langle C(s), v_2\rangle/\langle C(s), v\rangle$ and adding both, we obtain

$$\eta \frac{\langle B(s), v\rangle}{\langle C(s), v\rangle} - \gamma(\eta, s) < \eta x(s).$$

For each $(\eta, s) \in T\backslash W^*$ we have

$$\frac{\langle B(s), v_1\rangle}{\langle C(s), v_1\rangle} - \gamma(\eta, s) \leqslant \eta x(s) + K$$

and

$$\eta \frac{\langle B(s), v_2\rangle}{\langle C(s), v_2\rangle} - \gamma(\eta, s)\, z \leqslant \eta x(s) - M.$$

Proceeding as before, we obtain

$$\eta \frac{\langle B(s), v\rangle}{\langle C(s), v\rangle} - \gamma(\eta, s)\, z \leqslant \eta x(s) + K\rho \frac{\langle C(s), v_1\rangle}{\langle C(s), v\rangle} - M(1-\rho) \frac{\langle C(s), v_2\rangle}{\langle C(s), v\rangle}$$

$$\leqslant \eta x(s) + \frac{\langle C(s), v_2\rangle}{\langle C(s), v\rangle} [(K\alpha + M)\rho - M] < \eta x(s). \quad \blacksquare$$

3. Continuity Properties of the Feasible Set

Proposition 3.1. *The mapping* $Z: \mathfrak{M} \to \mathrm{POW}(S^{N-1} \times \mathbb{R})$ *is r-closed.*

Proof. Let the following sequences be given

$$\sigma_n := (B_n, C_n, \gamma_n, x_n) \text{ in } \mathfrak{M} \text{ and } (v_n, z_n) \in Z_{\sigma_n}$$

and elements

$$\sigma_0 := (B_0, C_0, \gamma_0, x_0) \text{ in } \mathfrak{M} \text{ and } (\bar{v}, \bar{z}) \in S^{N-1} \times \mathbb{R}$$

such that

$$\sigma_n \to \sigma_0 \text{ and } (v_n, z_n) \to (\bar{v}, \bar{z}) \text{ and } \bar{v} \in U_{\sigma_0}.$$

For each $n \in \mathbb{N}$ we have

$$\underset{(\eta, s) \in T}{\forall} \quad \eta \frac{\langle B_n(s), v_n \rangle}{\langle C_n(s), v_n \rangle} - \gamma_n(\eta, s)\, z_n \leqslant \eta x_n(s).$$

Since $\bar{v} \in U_{\sigma_0}$, we receive for $n \to \infty$

$$\underset{(\eta, s) \in T}{\forall} \quad \eta \frac{\langle B_0(s), \bar{v} \rangle}{\langle C_0(s), \bar{v} \rangle} - \gamma_0(\eta, s)\, \bar{z} \leqslant \eta x_0(s),$$

i.e., Z is r-closed. ∎

Remark. In general the mapping

$$Z: \mathfrak{M} \to \mathrm{POW}(S^{N-1} \times \mathbb{R})$$

is not closed in the usual sense, as the following consideration shows. Assume $N \geqslant 2$ and choose a parameter $\sigma = (B, C, \gamma, x)$ such that

$$\underset{(\eta, s) \in T}{\forall} \quad B(s) = 0 \qquad \text{and} \qquad \gamma(\eta, s) > 0.$$

Consequently, there exists $(v_0, z_0) \in Z_\sigma$. By Lemma 3.4, there exists an element w_0 in S^{N-1} such that

$$\underset{s \in S}{\forall} \langle C(s), w_0 \rangle \geqslant 0 \qquad \text{and} \qquad \underset{s_0 \in S}{\exists} \langle C(s_0), w_0 \rangle = 0.$$

Define $\sigma_n := \sigma$. Then we have

$$\sigma_n \to \sigma \quad \text{and} \quad \left(w_0 + \frac{1}{n} v_0, z_0 \right) \in Z_{\sigma_n} \quad \text{and} \quad \left(w_0 + \frac{1}{n} v_0, z_0 \right) \to (w_0, z_0).$$

Since $(w_0, z_0) \notin Z_\sigma$, Z is not closed in σ.

PROPOSITION 3.2. *Let σ_0 be an element in $\mathfrak{M}$. Then the following statements are equivalent*:

(1) *Z is lower semicontinuous in σ_0,*

(2) *F is lower semicontinuous in σ_0,*

(3) *E is upper semicontinuous in σ_0,*

(4) *σ_0 satisfies the Slater condition.*

Proof. (1)$\Rightarrow$(2). Assume F is not lower semicontinuous in σ_0. Then there exist an open set $W \subset C(S) \times \mathbb{R}$ and a sequence (σ_n) in $\mathfrak{M}$ such that

$$F_{\sigma 0} \cap W \neq \varnothing \qquad \text{and} \qquad \sigma_n \to \sigma_0 \qquad \text{and} \qquad \forall_{n \in \mathbb{N}} F_{\sigma_n} \cap W = \varnothing.$$

The mapping $A_{\sigma 0}: U_{\sigma 0} \times \mathbb{R} \to C(S) \times \mathbb{R}$ defined by

$$A_{\sigma 0}(v, z) := \left(\frac{\langle B_0, v \rangle}{\langle C_0, v \rangle}, z \right)$$

is continuous. Thus, the set

$$W_0 := A_{\sigma 0}^{-1}(W)$$

is an open subset of $U_{\sigma 0} \times \mathbb{R}$ and is also open in $S^{N-1} \times \mathbb{R}$. Obviously, we have

$$Z_{\sigma 0} \cap W_0 \neq \varnothing.$$

Choose an element (v_0, z_0) in $Z_{\sigma 0} \cap W_0$ and a compact neighborhood W_1 of (v_0, z_0), which is contained in W_0. Then we have also

$$Z_{\sigma 0} \cap W_1 \neq \varnothing.$$

Since Z is lower semicontinuous in σ_0 there exists an open neighborhood $W_2 \subset \mathfrak{M}$ of σ_0 such that

$$\forall_{\sigma \in W_2} Z_\sigma \cap W_1 \neq \varnothing.$$

For n large enough, say $n \geqslant n_0$, we have $\sigma_n \in W_2$. For each $n \geqslant n_0$, choose an element (v_n, z_n) in $Z_{\sigma_n} \cap W_1$. Since W_1 is compact and contained in $U_{\sigma 0} \times \mathbb{R}$, we can assume that (v_n, z_n) converges to some $(\bar{v}, \bar{z})$, which is contained in $U_{\sigma 0} \times \mathbb{R}$. By Proposition 3.1, $(\bar{v}, \bar{z}) \in Z_{\sigma 0}$. Further we have $(\bar{v}, \bar{z}) \in W_1 \subset W_0$. Consequently, we have $(\bar{v}, \bar{z}) \in W_0 \cap Z_\sigma$, which implies also $(\bar{r}, \bar{z}) \in W \cap F_{\sigma 0}$, where $\bar{r} := \langle B, \bar{v} \rangle / \langle C, \bar{v} \rangle$.

If we set

$$r_n := \frac{\langle B_n, v_n \rangle}{\langle C_n, v_n \rangle},$$

then we have $(r_n, z_n) \in F_{\sigma_n}$ and $(r_n, z_n) \to (\bar{r}, \bar{z})$. Thus, for n large enough, we have $(r_n, z_n) \in W$ which contradicts $F_{\sigma_n} \cap W = \emptyset$ for each $n \in \mathbb{N}$.

(2) $\Rightarrow$ (3). For $\varepsilon > 0$ define the open set

$$W_\varepsilon := \{(r, z) \in C(S) \times \mathbb{R} \mid |z - E_{\sigma_0}| < \varepsilon\}.$$

Since $W_\varepsilon \cap F_{\sigma_0} \neq \emptyset$, there exists an open neighborhood $W \subset \mathfrak{M}$ of σ_0 such that

$$\sigma \in W \Rightarrow \exists (r, z) \in F_\sigma \cap W_\varepsilon$$

which implies $E_\sigma - E_{\sigma_0} \leqslant z - E_{\sigma_0} < \varepsilon$.

(3) $\Rightarrow$ (4). Assume σ_0 does not satisfy Slater-condition. By Proposition 2.4, there exists a non-empty closed subset $T^* \subset T$ such that

$$\forall_{(v,z) \in Z_{\sigma_0}} \quad \forall_{(\eta, s) \in T^*} \quad \frac{\langle B_0(s), v \rangle}{\langle C_0(s), v \rangle} = x(s) \qquad \text{and} \qquad \gamma(\eta, s) = 0.$$

For $\eta \in \{-1, 1\}$ define the closed and disjoint sets

$$S_\eta := \{s \in S \mid (\eta, s) \in T^*\}.$$

By Urysohn's lemma there exists a continuous function $\Theta: S \to [-1, 1]$ such that

$$\forall_{s \in S_\eta} \quad \Theta(s) = \eta,$$

$\eta \in \{-1, 1\}$. Now define sequences

$$B_n := B_0, \qquad C_n := C_0, \qquad \gamma_n := \gamma_0 + \frac{1}{n}, \qquad x_n := x_0 - \frac{(1 + E_{\sigma_0})\,\Theta}{n},$$

$n \in \mathbb{N}$. The sequence

$$\sigma_n := (B_n, C_n, \gamma_n, x_n)$$

converges to σ_0 for $n \to \infty$.

For each $n \in \mathbb{N}$, we have $Z_{\sigma_n} \neq \emptyset$. In fact, choose an element $v_0 \in U_{\sigma_0}$ such that

$$(v_0, 1 + E_{\sigma_0}) \in Z_{\sigma_0}.$$

Then for each $(\eta, s) \in T$ we have the estimate

$$\begin{aligned}
&\eta \frac{\langle B_n(s), v_0 \rangle}{\langle C_n(s), v_0 \rangle} - \gamma_n(\eta, s)(1 + E_{\sigma_0}) \\
&\quad = \eta \frac{\langle B_0(s), v_0 \rangle}{\langle C_0(s), v_0 \rangle} - \gamma_0(\eta, s)(1 + E_{\sigma_0}) - \frac{(1 + E_{\sigma_0})}{n} \\
&\quad \leqslant \eta x_0(s) - \frac{\eta(1 + E_{\sigma_0})\, \Theta(s)}{n} + \frac{1 + E_{\sigma_0}}{n} [\eta \Theta(s) - 1] \\
&\quad \leqslant \eta x_n(s),
\end{aligned}$$

i.e., $(v_0, 1 + E_{\sigma_0}) \in Z_{\sigma_n}$ for each $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$ and each $(v, z) \in Z_{\sigma_n}$ we have the estimate $z \geqslant E_{\sigma_0} + 1$. In fact, by Proposition 2.4 (ii) there exists a point $(\eta_0, s_0) \in T^*$ such that

$$\eta_0 \left(\frac{\langle B_0(s_0), v \rangle}{\langle C_0(s_0), v \rangle} - x_0(s_0) \right) \geqslant 0 \quad \text{and} \quad \gamma_0(\eta_0, s_0) = 0 \quad \text{and} \quad \Theta(s_0) = \eta_0,$$

which implies

$$\begin{aligned}
z &\geqslant \frac{\eta_0(\langle B_0(s), v \rangle / \langle C_0(s), v \rangle - x_0(s)) + \eta_0 \Theta(s)(E_{\sigma_0} + 1)/\eta}{\gamma_0(\eta_0, s_0) + 1/n} \\
&\geqslant E_{\sigma_0} + 1.
\end{aligned}$$

Consequently, $E_{\sigma_n} = E_{\sigma_0} + 1$ contradicting the upper semicontinuity of E at σ_0.

(4) ⇒ (1). Let W be an open set such that

$$Z_{\sigma_0} \cap W \neq \varnothing.$$

By Proposition 2.1, we have

$$\overline{Z_{\sigma_0}^{<}} = Z_{\sigma_0}$$

which implies

$$Z_{\sigma_0}^{<} \cap W \neq \varnothing.$$

Choose $(v_0. z_0)$ in $Z_{\sigma_0}^{<} \cap W$. By Proposition 2.3, there exists an open neighborhood $W_0 \subset \mathfrak{M}$ of σ_0 such that

$$\sigma \in W \Rightarrow (v_0, z_0) \in Z_{\sigma}^{<},$$

i.e., Z is lower semicontinuous in σ_0. ∎

COROLLARY 3.3. *Let $\sigma_0 \in \mathfrak{M}$ satisfy the Slater condition. Then the mappings $\pi_1 \circ Z$ and $\pi_1 \circ F$ are lower semicontinuous.*

LEMMA 3.4. *Let $N \geqslant 2$ and $C \in C(S, \mathbb{R}^N)$ be such that $U_C \neq \varnothing$. Then there exists an element w in S^{N-1} such that*

$$\text{(a)} \quad \forall_{s \in S} \langle C(s), w \rangle \geqslant 0,$$

$$\text{(b)} \quad \exists_{s_0 \in S} \langle C(s_0), w \rangle = 0.$$

Proof. Let v_0 in $\mathbb{R}^N$ be such that

$$\forall_{s \in S} \langle C(s), v_0 \rangle > 0.$$

The assumption $N \geqslant 2$ implies that there exists an element w_0 in $\mathbb{R}^N$ such that v_0 and w_0 are linearly independent. For $\lambda > 0$ small enough we have

$$\forall_{s \in S} \langle C(s), v_0 + \lambda w_0 \rangle > 0.$$

Define $v_1 := v_0 + \lambda w_0$ and let $\beta \in \mathbb{R}$ and $s_0 \in S$ be given by

$$\beta := \frac{\langle C(s_0), v_0 \rangle}{\langle C(s_0), v_1 \rangle} := \min_{s \in S} \frac{\langle C(s), v_0 \rangle}{\langle C(s), v_1 \rangle}.$$

Then the element

$$w := \frac{v_0 - \beta v_1}{\| v_0 - \beta v_1 \|}$$

has the required properties. ∎

PROPOSITION 3.5. *Consider the mapping*

$$Z: \mathfrak{M} \to \mathrm{POW}(S^{N-1} \times \mathbb{R}).$$

Then we have

(i) *If $N = 1$, then Z is upper semicontinuous on $\mathfrak{M}$.*

(ii) *If $N \geqslant 2$, then, for all σ in $\mathfrak{M}$, the mapping Z is not upper semicontinuous at σ.*

Proof. (i) Let σ in $\mathfrak{M}$ be given. Then there exists an element v_0 in $\mathbb{R}$ such that

$$\| v_0 \| = 1 \qquad \text{and} \qquad \forall_{s \in S} \langle C(s), v_0 \rangle = C(s) \cdot v_0 > 0.$$

We can assume $v_0 = 1$. Then, for some $\alpha > 0$, we have

$$\forall_{s \in S} \; C(s) \geqslant \alpha > 0.$$

By way of contradiction, suppose Z is not upper semicontinuous in σ. Then there exists an open set W and a sequence (σ_n) in $\mathfrak{M}$ such that

$$Z_\sigma \subset W \qquad \text{and} \qquad \sigma_n \to 0 \qquad \text{and} \qquad \forall_{n \in \mathbb{N}} \; Z_{\sigma_n} \not\subset W.$$

Since $\sigma_n \to \sigma$ we have

$$\forall_{s \in S} \; C_n(s) \geqslant \frac{\alpha}{2},$$

for n large enough, which implies that only points of the form $(1, z_n)$ are contained in Z_{σ_n}. Thus, there exists an element $(1, z)$ in $Z_{\sigma_n} \setminus W$.

Since $\mathfrak{M} = \mathfrak{L}$, we have $(1, E_\sigma) \in Z_\sigma$, and consequently

$$Z_\sigma = \{(1, z) \in \mathbb{R}^2 \mid z \geqslant E_\sigma\}.$$

Then there exists an $\varepsilon > 0$ such that $E_\sigma < z + \varepsilon$ implies $(1, z) \in W$. Hence $(1, z_n) \notin W$ implies $z_n \leqslant E_\sigma - \varepsilon$ for n large enough. Then we have

$$\forall_{(\eta, s) \in T} \; \eta \left(\frac{B_n(s)}{C_n(s)} - x_n(s) \right) \leqslant \gamma_n(\eta, s)\, z_n$$
$$\leqslant \gamma_n(\eta, s)(E_\sigma - \varepsilon),$$

which implies

$$\forall_{(\eta, s) \in T} \; \eta \left(\frac{B(s)}{C(s)} - x(s) \right) \leqslant \gamma(\eta, s)(E_\sigma - \varepsilon),$$

i.e., $(1, E_\sigma - \varepsilon) \in Z_\sigma$ contradicting E_σ to be the minimum value.

(ii) Let σ in $\mathfrak{M}$ be given. By Lemma 3.4 there exists an element w in S^{N-1} with the properties (a) and (b). Define the sequence

$$\sigma_n := (B, C_n, \gamma_n, x)$$

by setting

$$C_n(s) := C(s) + \frac{w}{n} \qquad \text{and} \qquad \gamma_n(\eta, s) := \gamma(\eta, s) + \frac{1}{n}$$

for each $s \in S$, $(\eta, s) \in T$, and $n \in \mathbb{N}$. Since

$$\forall_{n \in \mathbb{N}} \; \forall_{s \in S} \; \langle C_n(s), w \rangle = \langle C(s), w \rangle + \frac{1}{n} \geqslant \frac{1}{n}$$

and

$$\underset{n\in\mathbb{N}}{\forall}\ \underset{(\eta,s)\in T}{\exists}\ \gamma_n(\eta,s)>0,$$

there exists, for each $n\in\mathbb{N}$, a real number z_n such that $(w,z_n)\in Z_{\sigma_n}$. The open set

$$W:=\left\{(v,z)\in S^{N-1}\times\mathbb{R}\ \middle|\ \underset{s\in S}{\forall}\ \langle C(s),v\rangle>0\right\}$$

contains Z_σ but not the element (w,z_n), $n\in\mathbb{N}$, because $\langle C(s_0),w\rangle=0$. Since $\sigma_n\to\sigma$, Z cannot be upper semicontinuous at σ. ∎

Choose $B, C\colon S\to\mathbb{R}^N$ such that for some γ, x, the parameter $\sigma=(B,C,\gamma,x)$ is contained in $\mathfrak{M}$. If we restrict the mapping

$$F\colon\mathfrak{M}\to\mathrm{POW}(C(S)\times\mathbb{R})$$

to the set

$$\mathfrak{M}_{B,C}:=\{(B,C,\gamma,x)\in\mathfrak{M}\},$$

then the continuous mapping A_σ defined in the proof of Proposition 3.2 is independent of σ. Then F has the factorization $F=A\circ Z$, and, by Proposition 3.5(ii) we obtain

PROPOSITION 3.6. *Consider the mapping*

$$F\colon\mathfrak{M}_{B,C}\to\mathrm{POW}(C(S)\times\mathbb{R}).$$

If $N=1$, then F is upper semicontinuous on $\mathfrak{M}_{B,C}$.

PROPOSITION 3.7. *For all $\sigma\in\mathfrak{M}$ such that the set V_σ is nowhere dense in $C(S)$, the mapping*

$$F\colon\mathfrak{M}\to\mathrm{POW}(C(S)\times\mathbb{R})$$

is not upper semi-continuous at σ.

Proof. Let $\sigma\in\mathfrak{M}$ be given and choose an element (w,z) in $U_\sigma\times\mathbb{R}$. Since V_σ is nowhere dense in $C(S)$, the set

$$M:=\bigcup_{n\in\mathbb{N}}\ [V_\sigma(n\langle C,w\rangle+1)]-n\langle B,w\rangle$$

is also nowhere dense in $C(S)$. Consequently, there exists a function $\Theta\notin M$.

Define a sequence $\sigma_n := (B_n, C_n, \gamma_n, x)$ by setting

$$B_n := B + \frac{\Theta w}{n}, \qquad C_n := C + \frac{w}{n}, \qquad \gamma_n := \gamma + \frac{1}{n}$$

for each $N \in \mathbb{N}$. Since

$$\forall_{n \in \mathbb{N}} \ \forall_{s \in S} \ \langle C_n(s), w \rangle = \langle C(s), w \rangle + \frac{1}{n} \geqslant \frac{1}{n}$$

and

$$\forall_{n \in \mathbb{N}} \ \forall_{(\eta, s) \in T} \ \gamma_n(\eta, s) > 0,$$

there exists, for each $n \in \mathbb{N}$, a real number z_n such that

$$(r_n, z_n) \in F_{\sigma_n},$$

where $r_n := \langle B_n, w \rangle / \langle C_n, w \rangle$. We can assume $z_n \to \infty$. Thus, the set $\{(r_n, z_n)\}$ has no limit point in $C(S) \times \mathbb{R}$ and consequently, it is closed in $C(S) \times \mathbb{R}$.

We claim

$$\forall_{n \in \mathbb{N}} \ (r_n, z_n) \notin F_\sigma.$$

In fact, we have

$$\forall_{s \in S} \ \langle r_n(s)\, C_n(s) - B_n(s), w \rangle = 0$$

which implies

$$\forall_{s \in S} \ \Theta(s) = r_n(s)[n \langle C(s), w \rangle + 1] - n \langle B(s), w \rangle.$$

By definition of Θ, the function r_n cannot be contained in V_σ, which proves the claim.

The open set

$$W := C(S) \times \mathbb{R} \setminus \{(r_n, z_n)\}$$

contains F_σ but not the elements (r_n, z_n), $n \in \mathbb{N}$. Thus, we have

$$F_{\sigma_n} \not\subset W.$$

Since $\sigma_n \to \sigma$, the mapping F cannot be upper semicontinuous at σ. ∎

4. Continuity Properies of the Minimal Value

Proposition 4.1. *Let $N=1$. Then $E: \mathfrak{M} \to \mathbb{R}$ is continuous in $\sigma_0 \in \mathfrak{M}$ if and only if σ_0 satisfies the Slater condition.*

Proof. Assume σ_0 satisfies the Slater condition. We claim that E is lower semicontinuous in σ_0. In fact, define for $\varepsilon > 0$ the open set

$$W_\varepsilon := \{(v, z) \in S^{N-1} \times \mathbb{R} \mid E_{\sigma_0} - z < \varepsilon\},$$

which contains Z_{σ_0}. By Proposition 3.5, Z is upper semicontinuous at σ_0. Hence there exists an open neighborhood $W \subset \mathfrak{M}$ of σ_0 such that

$$\sigma \in W \Rightarrow Z_\sigma \subset W_\varepsilon,$$

which implies

$$\underset{\sigma \in W}{\forall} \; E_{\sigma_0} - E_\sigma < \varepsilon$$

and proves the claim. Since, by Proposition 3.2, E is also upper semicontinuous at σ_0, the continuity of E at σ_0 follows.

Now assume E is continuous in σ_0. Then E is also upper continuous at σ_0 and, by Proposition 3.2, σ_0 satisfies the Slater condition. ∎

Proposition 4.2. *Let $N \geqslant 2$ and $\sigma_0 \in \mathfrak{L}$. Consider the statements*

(1) *$P: \mathfrak{L} \to \mathrm{POW}(S^{N-1} \times \mathbb{R})$ is upper semicontinuous at σ_0,*

(2) *$E: \mathfrak{L} \to \mathrm{POW}(S^{N-1} \times \mathbb{R})$ is continuous at σ_0,*

(3) *σ_0 satisfies the Slater condition.*

Then we have the implications and the converse implications are

$$(1) \Rightarrow (2) \Rightarrow (3)$$

not true.

Proof $(1) \Rightarrow (2)$. For $\varepsilon > 0$ define the open set

$$W_\varepsilon := \{(v, z) \in S^{N-1} \times \mathbb{R} \mid \; |E_{\sigma_0} - z| < \varepsilon\},$$

which contains P_{σ_0}. Since P is upper semicontinuous in σ_0, there exists an open neighborhood $W \subset \mathfrak{L}$ of σ_0 such that

$$\sigma \in W \Rightarrow P_\sigma \subset W_\varepsilon,$$

which implies $|E_\sigma - E_{\sigma_0}| < \varepsilon$, i.e., the continuity of E at σ_0.

(2) ⇒ (3). The assumption implies that E is also upper semicontinuous at σ_0. Then (3) follows from Proposition 3.2.

(2) *does not imply* (1). Let $S = [-1, 1]$, $N = 3$, and define $\sigma := (B, C, \gamma, x)$ by setting

$$B(s) := (1, s, s^2) \qquad C(s) := (1, s, s^3),$$
$$\gamma(\eta, s) := 1, \qquad x(s) := 1 + \sin(2\pi s).$$

We claim that the minimal set P_σ is given by

$$P_\sigma = \{(v, 1) \in Z_\sigma \mid v_3 = 0\}.$$

In fact, we have for each $(v, 1) \in P_\sigma$

$$r_0 := \frac{\langle B, v\rangle}{\langle C, v\rangle} = 1$$

and consequently

$$\underset{(\eta, s) \in T}{\forall} \eta \left(\frac{\langle B, v\rangle}{\langle C, v\rangle} - 1 - \sin(2\pi s)\right) = -\eta \sin(2\pi s) \leqslant 1$$

with the active points

$$(-1, -\tfrac{3}{4}), (1, -\tfrac{1}{4}), (-1, \tfrac{1}{4}), (1, \tfrac{3}{4}),$$

which implies $E_\sigma \leqslant 1$. Consider a point $(\bar{v}, \hat{E}) \in Z_\sigma$ such that $\bar{v}_3 \neq 0$. Since

$$\underset{s \in S}{\forall} \langle C(s), \bar{v}\rangle > 0,$$

we have $\bar{v}_1 \neq 0$ and, consequently,

$$\frac{\bar{v}_1 + \bar{v}_2 s + \bar{v}_3 s^2}{\bar{v}_1 + \bar{v}_2 s + \bar{v}_3 s^3} = 1 + \frac{\bar{v}_3(s^2 - s^3)}{\bar{v}_1 + \bar{v}_2 s + \bar{v}_3 s^3}.$$

In the open interval $(0, 1)$ the expression

$$\frac{s^2 - s^3}{\bar{v}_1 + \bar{v}_2 s + \bar{v}_3 s^3}$$

is always positive. If $\bar{v}_3 > 0$, then we have for $\eta_0 = 1$ and $s_0 = \frac{3}{4}$ the estimate

$$\hat{E} \geqslant 1 \cdot \left(1 + \frac{\bar{v}_3(s_0^2 - s_0^3)}{\bar{v}_1 + \bar{v}_2 s_0 + \bar{v}_3 s_0^3} - 1 - \sin(2\pi s_0)\right)$$
$$= \frac{\bar{v}_3(s_0^2 - s_0^3)}{\bar{v}_1 + \bar{v}_2 s_0 + \bar{v}_3 s_0^3} + 1 > 1.$$

Similarly, we obtain for $\bar{v}_3 < 0$ the estimate $\hat{E} > 1$. Thus, it follows, that for a solution (v, E_0) we have $v_3 = 0$, which proves the claim.

Then the sequence $(v_n, 1)$ in P_σ, where

$$v_n = (\sqrt{1-\alpha_n^2}, \alpha_n, 0) \qquad \text{and} \qquad \alpha_n := \frac{1}{\sqrt{2}} - \frac{1}{n}$$

converges to $(1/\sqrt{2}, 1/\sqrt{2}, 0, 1)$, which does not belong to P_σ. Consequently, P_σ is not compact and, by Proposition 5.3, P is not upper semicontinuous at σ.

However, E is continuous at σ. In fact, consider a sequence (σ_n) in $\mathfrak{L}$, which converges to σ_0. Choose points (v_n, E_{σ_n}) in P_{σ_n}. Since $\gamma > 0$, by Proposition 3.2, E is upper semicontinuous at σ. Thus, the sequence (v_n, E_{σ_n}) is bounded. Consider any convergent subsequence of (v_n, E_{σ_n}) (again denoted by (v_n, E_{σ_n})), with limit $(\bar{v}, \hat{E})$. By upper semicontinuity of E at σ, we have $\hat{E} \leqslant E_\sigma$. The element $\bar{v}$ satisfies the inequality

$$\underset{s \in S}{\forall} \langle C(s), \bar{v} \rangle \geqslant 0,$$

and we have $\|\bar{v}\| = 1$. Thus, the polynomial $\langle C(s), \bar{v} \rangle$ can have at most one zero (not counting multiplicities) in the open interval $(0, 1)$ and, consequently, there exists an active point (η_0, s_0) which is different from this zero. Choose an element (v_0, E_σ) in P_σ. By Lemma 4.3, the element

$$v_\varepsilon := (1-\varepsilon)\,\bar{v} + \varepsilon v_0$$

satisfies for $0 < \varepsilon \leqslant 1$ and for each $(\eta, s) \in \Gamma$ the inequalities

$$\langle C(s), v_\varepsilon \rangle > 0$$

and

$$\eta\left(\frac{\langle Bs), v_\varepsilon \rangle}{\langle C(s), v_\varepsilon \rangle} - x(s)\right) \leqslant \gamma(\eta, s)\left[(1-\varepsilon)\frac{\langle C(s), \bar{v} \rangle}{\langle C(s), v_\varepsilon \rangle}\hat{E} + \varepsilon\frac{\langle C(s), v_0 \rangle}{\langle C(s), v_\varepsilon \rangle}E_\sigma\right],$$

which imply $(v_\varepsilon/\|v_\varepsilon\|, E_\sigma) \in P_\sigma$. For (η_0, s_0) we have

$$E_\sigma = \eta_0\left(\frac{\langle B(s_0), v_\varepsilon)}{\langle C(s_0), v_\varepsilon \rangle} - x(s_0)\right)$$

$$\leqslant 1 \cdot \left[(1-\varepsilon)\frac{\langle C(s_0), \bar{v} \rangle}{\langle C(s_0), v_\varepsilon \rangle}\hat{E} + \varepsilon\frac{\langle C(s_0), v_0 \rangle}{\langle C(s_0), v_\varepsilon \rangle}E_\sigma\right],$$

which implies

$$(1-\varepsilon)\frac{\langle C(s_0), \bar{v} \rangle}{\langle C(s_0), v_\varepsilon \rangle}E_\sigma \leqslant (1-\varepsilon)\frac{\langle C(s_0), \bar{v} \rangle}{\langle C(s_0), v_\varepsilon \rangle}\hat{E}$$

or $E_\sigma \leqslant \hat{E}$, and consequently $E_\sigma = \hat{E}$.

Since we have considered an arbitrary convergent subsequence of (v_n, E_{σ_n}), the sequence E_{σ_n} converges to E_σ, i.e., E is continuous at σ.

(3) *does not imply* (2). Let $S = \{0\}$, $N = 2$, and define

$$B(0) := (0, 0), \qquad C(0) := (0, 1), \qquad x(0) := 1, \qquad \gamma(\eta, 0) := \frac{1-\eta}{2}.$$

Then we have

$$Z_\sigma = \{(v, z) \in S^1 \times \mathbb{R} \mid v_2 > 0 \text{ and } z \geqslant 1\},$$

$E_\sigma = 1$, and $P_\sigma = \{(v, z) \in Z_\sigma \mid z = 1\}$. Any $(v, z) \in Z_\sigma$ with $z > 1$ is a Slater element.

Define a sequence (σ_n) by

$$B_n(0) := \left(\frac{1}{n}, 0\right), \qquad C_n(0) := (0, 1), \qquad x_n(0) := 1, \qquad \gamma_n(\eta, 0) := \frac{1-\eta}{2}.$$

Then we have

$$Z_{\sigma_n} = \left\{(v, z) \in S^1 \times \mathbb{R} \mid v_2 > 0 \text{ and } \frac{v_1}{nv_2} \leqslant 1 \text{ and } z \geqslant 1 - \frac{v_1}{nv_2}\right\},$$

$$E_{\sigma_n} = 0,$$

and

$$P_{\sigma_n} = \left\{\left(\frac{n}{\sqrt{n^2+1}}, \frac{1}{\sqrt{n^2+1}}, 0\right)\right\}.$$

It is clear that $\sigma_n \to \sigma$ but $E_{\sigma_n} \nrightarrow E_\sigma$. ∎

Remark. A similar proof to (1) ⇒ (2) shows also that the condition

> (1a) $Q: \mathfrak{L} \to \text{POW}(C(S) \times \mathbb{R})$ is upper semicontinuous at σ_0, implies condition (2).

The implication (2) ⇒ (3) is also true for $\sigma_0 \in \mathfrak{M}$.

LEMMA 4.3. *Let there be given a sequence (σ_n) in $\mathfrak{L}$ and elements $(w_n, z_n) \in Z_{\sigma_n}$, $\sigma \in \mathfrak{L}$, $(w_0, \hat{E}) \in S^{N-1} \times \mathbb{R}$ such that*

$$\sigma_n \to \sigma \qquad \text{and} \qquad (w_n, z_n) \to (w_0, \hat{E}).$$

If $(w_0, z_0) \in Z_0$ and $0 < \varepsilon \leqslant 1$, then the element

$$v_\varepsilon := (1-\varepsilon)\, w_0 + \varepsilon v_0$$

satisfies for each $(\eta, s) \in T$ *the inequalities*

$$\langle C(s), v_\varepsilon \rangle > 0$$

and

$$\eta\left(\frac{\langle B(s), v_\varepsilon \rangle}{\langle C(s), v_\varepsilon \rangle} - x(s)\right) \leqslant \gamma(\eta, s)\left[(1-\varepsilon)\frac{\langle C(s), w_0 \rangle}{\langle C(s), v_\varepsilon \rangle}\hat{E} + \varepsilon\frac{\langle C(s), v_0 \rangle}{\langle C(s), v_\varepsilon \rangle} z_0\right].$$

Proof. For $n \in \mathbb{N}$ define the elements

$$v_\varepsilon^n := (1-\varepsilon) w_n + \varepsilon v_0.$$

Since $\sigma_n \to \sigma$, for n large enough we have

$$\forall_{s \in S} \langle C_n(s), v_0 \rangle \geqslant \tfrac{1}{2} \min_{\bar{s} \in S} \langle C(\bar{s}), v_0 \rangle > 0,$$

which implies

$$\frac{v_\varepsilon^n}{\|v_\varepsilon^n\|} \in U_{\sigma_n} \qquad \text{and} \qquad \frac{v_\varepsilon}{\|v_\varepsilon\|} \in U_\sigma.$$

For each $(\eta, s) \in T$ we have the estimate

$$\begin{aligned}
&\eta\left(\frac{\langle B_n(s), v_\varepsilon^n \rangle}{\langle C_n(s), v_\varepsilon^n \rangle} - x_n(s)\right) \\
&\quad = (1-\varepsilon)\frac{\langle C_n(s), w_n \rangle}{\langle C_n(s), v_\varepsilon^n \rangle}\cdot\eta\left[\frac{\langle B_n(s), w_n \rangle}{\langle C_n(s), w_n \rangle} - x_n(s)\right] \\
&\qquad + \varepsilon\frac{\langle C_n(s), v_0 \rangle}{\langle C_n(s), v_\varepsilon^n \rangle}\cdot\eta\left[\frac{\langle B_n(s), v_0 \rangle}{\langle C_n(s), v_0 \rangle} - x_n(s)\right] \\
&\quad \leqslant (1-\varepsilon)\frac{\langle C_n(s), w_n \rangle}{\langle C_n(s), v_E^n \rangle}\gamma_n(\eta, s)\, z_n \\
&\qquad + \varepsilon\frac{\langle C_n(s), v_0 \rangle}{\langle C_n(s), v_\varepsilon^n \rangle}\eta\left[\frac{\langle B_n(s), v_0 \rangle}{\langle C_n(s), v_0 \rangle} - x_n(s)\right].
\end{aligned}$$

For $n \to \infty$ we obtain

$$\begin{aligned}
&\eta\left(\frac{\langle B(s), v_\varepsilon \rangle}{\langle C(s), v_\varepsilon \rangle} - x_0(s)\right) \\
&\quad \leqslant (1-\varepsilon)\frac{\langle C(s), w_0 \rangle}{\langle C(s), v_\varepsilon \rangle}\gamma(\eta, s)\,\hat{E} + \varepsilon\frac{\langle C(s), v_0 \rangle}{\langle C(s), v_\varepsilon \rangle}\eta\left[\frac{\langle B(s), v_0 \rangle}{\langle C(s), v_0 \rangle} - x(s)\right] \\
&\quad \leqslant \gamma(\eta, s)\left[(1-\varepsilon)\frac{\langle C(s), w_0 \rangle}{\langle C(s), v_\varepsilon \rangle}\hat{E} + \varepsilon\frac{\langle C(s), v_0 \rangle}{\langle C(s), v_\varepsilon \rangle} z_0\right].
\end{aligned}$$

PROPOSITION 4.4. *Let $\sigma \in \mathfrak{L}$ be such that σ satisfies the Slater condition and P_σ is compact. Then E is continuous at σ.*

Proof. Let $(\sigma_n) \subset \mathfrak{L}$, $\sigma_n \to \sigma$, and consider the sequence (E_{σ_n}) in $\mathbb{R}$. Since, by Proposition 3.2, E is upper semicontinuous at σ, for each $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that

$$n \geqslant n_0 \Rightarrow E_{\sigma_n} - E_{\sigma_0} \leqslant \varepsilon.$$

Thus, the sequence (E_{σ_n}) is bounded and it suffices to prove that every convergent subsequence of (E_{σ_n}) (again denoted by (E_{σ_n})) converges to E_σ. We can also assume that there exist elements $(w_n, E_{\sigma_n}) \in P_{\sigma_n}$ such that (w_n) converges to an element $w_0 \in S^{N-1}$ and E_{σ_n} converges to $\hat{E}$. Then we have $\hat{E} \leqslant E_\sigma$.

Choose $(v_0, E_\sigma) \in P_\sigma$. By Lemma 4.3, for each $0 < \varepsilon \leqslant 1$, the element $v_\varepsilon := (1-\varepsilon)\, w_0 + \varepsilon v_0$ satisfies for each $(\eta, s) \in T$ the inequalities

$$\langle C(s), v_\varepsilon \rangle > 0$$

and

$$\eta \left(\frac{\langle B(s), v_\varepsilon \rangle}{\langle C(s), v_\varepsilon \rangle} - x(s) \right) \leqslant \gamma(\eta, s) \left[(1-\varepsilon) \frac{\langle C(s), w_0 \rangle}{\langle C(s), v_\varepsilon \rangle} \hat{E} + \varepsilon \frac{\langle C(s), v_0 \rangle}{\langle C(s), v_\varepsilon \rangle} E_\sigma \right],$$

which imply $(v_\varepsilon / \| v_\varepsilon \|, E_\sigma) \in P_\sigma$.

Define for each $m \in \mathbb{N}$ the element

$$v_m := \left(1 - \frac{1}{m}\right) w_0 + \frac{1}{m} v_0.$$

Since $(v_m / \| v_m \|, E_\sigma) \in P_\sigma$ and P_σ is compact, there exists a subsequence of $(v_m / \| v_m \|)$ (again denoted by $(v_m / \| v_m \|)$) and an element $(\bar{v}, E_\sigma) \in P_\sigma$ such that $v_m / \| v_m \| \to \bar{v}$. Since $\| v_m \| \to 1$ we also have $v_m \to \bar{v}$. Since $v_m \to w_0$, we have $\bar{v} = w_0$. Then the estimate

$$\forall_{(\eta, s) \in T}\ \eta \left(\frac{\langle B(s), v_m \rangle}{\langle C(s), v_m \rangle} - x(s) \right)$$
$$\leqslant \gamma(\eta, s) \left[\left(1 - \frac{1}{m}\right) \frac{\langle C(s), w_0 \rangle}{\langle C(s), v_m \rangle} \hat{E} + \frac{1}{m} \frac{\langle C(s), v_0 \rangle}{\langle C(s), v_m \rangle} E_\sigma \right]$$

implies, for $m \to \infty$,

$$\forall_{(\eta, s) \in T}\ \eta \left(\frac{\langle B(s), w_0 \rangle}{\langle C(s), w_0 \rangle} - x(s) \right) \leqslant \gamma(\eta, s)\, E,$$

which shows $E_\sigma \leqslant \hat{E}$. ∎

5. Continuity Properties of P

PROPOSITION 5.1. *If σ in $\mathfrak{L}$ satisfies the Slater condition, then the mapping*

$$P: \mathfrak{L} \to \mathrm{POW}(S^{N-1} \times \mathbb{R})$$

is r-closed in σ.

Proof. Let there be given sequences

$$\sigma_n := (B_n, C_n, \gamma_n, x_n) \text{ in } \mathfrak{L} \text{ and } (v_n, E_{\sigma_n}) \in P_{\sigma_n}$$

and $(\bar{v}, \bar{z}) \in S^{N-1} \times \mathbb{R}$ such that

$$\sigma_n \to \sigma \quad \text{and} \quad (v_n, E_{\sigma_n}) \to (\bar{v}, \bar{z}) \quad \text{and} \quad \bar{v} \in U_\sigma.$$

By Proposition 3.1, $(\bar{v}, \bar{z}) \in Z_\sigma$ and consequently, $\bar{z} \geqslant E_\sigma$.

Choose an element (v, E_σ) in P_σ. By Proposition 3.2, Z is lower semicontinuous in σ. Thus, there exists a subsequence of (σ_n) (again denoted by (σ_n)) and a sequence (w_n, z_n) in Z_{σ_n} such that

$$(w_n, z_n) \to (v, E_\sigma).$$

Then we have $E_{\sigma_n} \leqslant z_n$, which implies $\bar{z} \leqslant E_\sigma$, and thus $(\bar{v}, E_\sigma) \in P_\sigma$. ∎

LEMMA 5.2. *Assume $\sigma \in \mathfrak{L}$ satisfies the Slater condition and $(v_0, E_\sigma) \in P_\sigma$. Then for each $\lambda > 1$ the parameter σ_λ satisfies the Slater condition and the element $(v_0, \lambda E_\sigma)$ is contained in P_{σ_λ}, where*

$$\sigma_\lambda := (B, C, \gamma, x_\lambda)$$

and

$$x_\lambda := \frac{\langle B, v_0 \rangle}{\langle C, v_0 \rangle} + \lambda \left(x - \frac{\langle B, v_0 \rangle}{\langle C, v_0 \rangle} \right).$$

Proof. For each (v, z) in Z_σ and for each (η, s) in T we have

$$\eta \left(\frac{\langle B(s), v \rangle}{\langle C(s), v \rangle} - x_\lambda(s) \right)$$
$$+ \eta \left[\left(\frac{\langle B(s), v \rangle}{\langle C(s), v \rangle} - x(s) \right) + (\lambda - 1) \left(\frac{\langle B(s), v_0 \rangle}{\langle C(s), v_0 \rangle} - x(s) \right) \right]$$
$$\leqslant \gamma(\eta, s)\, z + (\lambda - 1)\, \gamma(\eta, s)\, E_\sigma \leqslant \gamma(\eta, s)\, \lambda z,$$

which implies $(v, \lambda z) \in Z_{\sigma_\lambda}$. If (v, z) is a Slater-element of Z_σ, then (v, λ, z) is a Slater-element of Z_{σ_λ}, i.e., $Z^<_{\sigma_\lambda} \neq \varnothing$, for each $\lambda > 1$.

If we consider the element $(v_0, E_\sigma) \in P_\sigma$, then we have for all $(\eta, s) \in M(\sigma, v_0, E_\sigma)$

$$\eta\left(\frac{\langle B(s), v_0\rangle}{\langle C(s), v_0\rangle} - x_\lambda(s)\right) = \eta\lambda\left(\frac{\langle B(s), v_0\rangle}{\langle C(s), v_0\rangle} - x(s)\right)$$
$$= \lambda\gamma(\eta, s)\, E_\sigma,$$

which implies

$$M(\sigma, v_0, E_\sigma) \subset M(\sigma_\lambda, v_0, \lambda E_\sigma).$$

By Proposition 2.2, $(v_0, \lambda E_\sigma) \in P_{\sigma_\lambda}$. ∎

PROPOSITION 5.3. *Assume* $\# S \geqslant N-1$ *and define the set*

$$\tilde{\mathfrak{L}} := \{\sigma \in \mathfrak{L} \mid P_\sigma \textit{ compact and } \sigma \textit{ satisfies the Slater condition}\}.$$

Then:

(i) *The mapping* $P\colon \mathfrak{L} \to \mathrm{POW}(S^{N-1} \times \mathbb{R})$ *is upper semicontinuous at* $\sigma \in \mathfrak{L}$ *if and only if* $\sigma \in \tilde{\mathfrak{L}}$;

(ii) *The set* $\tilde{\mathfrak{L}}$ *is open in* $\mathfrak{L}$.

Proof. (i), (1) Let P_σ be compact and σ satisfy the Slater condition. Suppose P is not upper semicontinuous at σ. Then there exists an open set W and sequences

$$\sigma_n \in \mathfrak{L} \qquad \text{and} \qquad (w_n, E_{\sigma_n}) \in P_{\sigma_n}$$

such that

$$P_\sigma \subset W \qquad \text{and} \qquad \sigma_n \to \sigma \qquad \text{and} \qquad (w_n, E_{\sigma_n}) \notin W.$$

By Proposition 4.4, $E_{\sigma_n} \to E_\sigma$. We can assume that $w_n \to w_0$ for some $w_0 \in S^{N-1}$. Since $(w_n, E_{\sigma_n}) \notin W$ we have $(w_0, E_\sigma) \notin P_\sigma$. By compactness of P_σ there exists a δ-neighborhood

$$\bigcup_{(v, E_\sigma) \in P_\sigma} \{(w, z) \in S^{N-1} \times \mathbb{R} \mid \|(w, z) - (v, E_\sigma)\| < \delta\},$$

which also does not contain (w_0, E_σ), hence

$$\underset{(v, E_\sigma) \in P_\sigma}{\forall} \|w_0 - v\| \geqslant \delta > 0.$$

Choose an element (v_0, E_σ) in P_σ. By Lemma 4.3, for $0 < \varepsilon \leqslant 1$, the element

$$v_\varepsilon := (1-\varepsilon)\, w_0 + \varepsilon v_0$$

satisfies for each $(\eta, s) \in T$ the inequalities

$$\langle C(s), v_\varepsilon \rangle > 0$$

and

$$\eta \left(\frac{\langle B(s), v_\varepsilon \rangle}{\langle C(s), v_\varepsilon \rangle} - x(s) \right) \leqslant \gamma(\eta, s) E_\sigma,$$

i.e., $(v_\varepsilon / \| v_\varepsilon \|, E_\sigma) \in P_\sigma$. Then, for $0 < \varepsilon \leqslant 1$, we have

$$\left\| w_0 - \frac{v_\varepsilon}{\| v_\varepsilon \|} \right\| \geqslant \delta > 0.$$

Since $v_\varepsilon / \| v_\varepsilon \| \to w_0$ at $\varepsilon \to 0$, we have a contradiction.

(i), (2). *Case* 1: $x \notin V_\sigma$. Let P be upper semicontinuous at σ. By Proposition 4.2, the parameter σ satisfies the Slater condition. Suppose P_σ is not compact. Then there exists a sequence of points (v_n, E_σ) in P_σ without a limit point in P_σ and, consequently, without a limit point in $U_\sigma \times \mathbb{R}$. For $n \in \mathbb{N}$, define

$$\lambda_n := 1 + \frac{1}{n}, \qquad r_n := \frac{\langle B, v_n \rangle}{\langle C, v_n \rangle},$$

$$x_n := r_n + \lambda_n (x - r_n), \qquad \sigma_n := (B, C, \gamma, x_n).$$

By Lemma 5.2, $(v_n, \lambda_n E_\sigma) \in P_{\sigma_n}$. The assumption $x \notin V_\sigma$ implies $E_\sigma > 0$. Thus, we have $(v_n, \lambda_n E_\sigma) \notin P_\sigma$. Consider the open set

$$W := (U_\sigma \times \mathbb{R}) \setminus \{(v_n, \lambda_n E_\sigma)\} \subset S^{N-1} \times \mathbb{R}.$$

Then we have $P_\sigma \subset W$ and $P_{\sigma_n} \not\subset W$ for each $n \in \mathbb{N}$.

Since

$$\begin{aligned} \| \sigma_n - \sigma \| &= \| x_n - x \|_\infty \\ &= (\lambda_n - 1) \| r_n - x \|_\infty \\ &\leqslant \| \gamma \|_\infty E_\sigma (\lambda_n - 1) \\ &= \frac{1}{n} \| \gamma \|_\infty E_\sigma, \end{aligned}$$

it follows that $\sigma_n \to \sigma$, which contradicts the upper semicontinuity of P at σ.

Case 2: $x \in V_\sigma$. In this case we have

$$P_\sigma = \left\{ (v, 0) \in U_\sigma \times \mathbb{R} \;\middle|\; x = \frac{\langle B, v \rangle}{\langle C, v \rangle} \right\}$$

and we will use the notation $r := x$. Proposition 4.2 implies that σ satisfies the Slater condition. If σ is normal, then, by Corollary 6.2, P_σ is compact. Thus, we can assume $\dim H_r \geqslant 2$.

Suppose, by way of contradiction, P_σ is not compact. Then there exists a sequence $(v_n, 0)$ in P_σ without a limit point, i.e., the set $\{(v_n, 0)\}$ is closed in P_σ and in view of Proposition 5.1 also closed in $U_\sigma \times \mathbb{R}$. Consider the linear space

$$\mathfrak{L}(r) := \{\langle rC - B, w\rangle \in C(S) \mid w \in \mathbb{R}^N\}.$$

If $\dim \mathfrak{L}(r) = 0$, then we have $V_\sigma = \{r\}$. Choose $\eta_0 \in \{-1, 1\}$ such that

$$\Theta(s) := \eta_0 \gamma(\eta_0, s)$$

is not the zero function. Define a sequence of parameters $\sigma_n := (B, C, \gamma, x_n)$ by setting

$$x_n := r - \frac{\Theta}{n}.$$

Then we have also $V_{\sigma_n} = \{r\}$. We claim, that $Q_{\sigma_n} = \{(r, 1/n)\}$. In fact, consider for each $(\eta, s) \in T$ the inequality

$$\eta(r(s) - x_n(s)) = \frac{\eta\eta_0\gamma(\eta_0, s)}{n} \leqslant \gamma(\eta, s)\frac{1}{n},$$

where we have equality for those (η_0, s) such that $\gamma(\eta_0, s) > 0$, i.e., $E_{\sigma_n} = 1/n$. Then $(v_n, 1/n)$ belongs to P_{σ_n}. Define the open set

$$W := U_\sigma = \mathbb{R} \Big\backslash \left\{\left(v_n, \frac{1}{n}\right)\right\},$$

which contains P_σ and does not contain P_{σ_n} for each $n \in \mathbb{N}$. Since $\sigma_n \to \sigma$ we have a contradiction to the upper semicontinuity of P at σ. Thus, P_σ is compact in this case.

If $\dim \mathfrak{L}(r) > 0$ choose a basis $\varphi_1, \varphi_2, ..., \varphi_d$ of $\mathfrak{L}_r$. Using the formula

$$\dim \mathfrak{L}_r + \dim H_r = N$$

(compare [8, Section 4]) and the estimate $\dim H_r \geqslant 2$, we have

$$d := \dim \mathfrak{L}_r \leqslant N - 2.$$

By assumption S contains at least $N-1$ points. Then there exist $1 \leqslant k \leqslant d+1$ points

$$s_1, s_2, ..., s_k \in S$$

such that the vectors

$$J(s_\kappa) := (\varphi_1(s_\kappa), \varphi_2(s_\kappa), ..., \varphi_d(s_\kappa)),$$

$\kappa = 1, 2, ..., k$ are linearly dependent. Thus, we can find real numbers $\lambda_1, \lambda_2, ..., \lambda_k$ such that

$$\sum_{\kappa=1}^{k} \lambda_\kappa J(s_\kappa) = 0.$$

We can assume that $\lambda_\kappa \neq 0$, $\kappa = 1, 2, ..., k$ and

$$\sum_{\kappa=1}^{k} |\lambda_\kappa| = 1.$$

Then the set

$$\{\text{sgn } \lambda_\kappa, s_\kappa) \in T \mid 1 \leqslant \kappa \leqslant k\}$$

is a critical set with respect to r (for the definitions compare B. Brosowski and C. Guerreiro [10]).

Define the disjoint and closed sets

$$S^+ := \{s_\kappa \in S \mid \text{sgn } \lambda_\kappa = 1 \text{ and } \gamma(\text{sgn } \lambda_\kappa, s_\kappa) > 0\}$$

and

$$S^- := \{s_\kappa \in S \mid \text{sgn } \lambda_\kappa = -1 \text{ and } \gamma(\text{sgn } \lambda_\kappa, s_\kappa) > 0\}.$$

We can assume that at least one of the sets S^+ and S^- is non-empty, replacing, if necessary, $\lambda_1, \lambda_2, ..., \lambda_k$ by $-\lambda_1, -\lambda_2, ..., -\lambda_k$ and using the condition

$$\forall_{s \in S} \; \gamma(1, s) + \gamma(-1, s) > 0.$$

By Urysohn's lemma there exist continuous functions $\Theta^+, \Theta^- : S \to [0, 1]$ such that

$$\Theta^+(s) := \begin{cases} 1 & \text{if } s \in S^+ \\ 0 & \text{if } s \in S^- \cup S^0 \end{cases}$$

and

$$\Theta^-(s) := \begin{cases} 1 & \text{if } s \in S^- \\ 0 & \text{if } s \in S^+ \cup S^0, \end{cases}$$

where

$$S_0 := \{s_\kappa \in S \mid \gamma(\operatorname{sgn} \lambda_\kappa, s_\kappa) = 0\}.$$

The function

$$\Theta(s) := \Theta^+(s)\,\gamma(1, s) - \Theta^-(s)\,\gamma(-1, s)$$

satisfies the inequalities

$$\underset{s \in S}{\forall} \quad -\gamma(-1, s) \leqslant \Theta(s) \leqslant \gamma(1, s)$$

and, consequently,

$$\underset{(\eta, s) \in T}{\forall} \quad \eta\Theta(s) \leqslant \gamma(\eta, s).$$

Define a sequence of parameters $\sigma_n := (B, C, \gamma, x_n)$ by setting

$$x_n := r - \frac{\delta\Theta}{n},$$

where $\delta > 0$ is chosen so small, that each σ_n satisfies the Slater condition.

We claim that $(r, \delta/n)$ is contained in Q_{σ_n}. In fact, consider for each $(\eta, s) \in T$ the inequality

$$\eta(r(s) - x_n(s)) = \frac{\delta}{n}\,\eta\Theta(s) \leqslant \gamma(\eta, s)\,\frac{\delta}{n}$$

with equality at least for the points

$$(\operatorname{sgn} \lambda_1, s_1), (\operatorname{sgn} \lambda_2, s_2), \ldots, (\operatorname{sgn} \lambda_\kappa, s_\kappa).$$

Since this set is critical with respect to r, by [10, Theorem 1.1], the result follows.

Define the open set

$$W := U_\sigma \times \mathbb{R} \Big\backslash \left\{\left(v_n, \frac{\delta}{n}\right)\right\},$$

which contains P_σ and does not contain P_{σ_n} for each $n \in \mathbb{N}$. Since $\sigma_n \to \sigma$ this contradicts the upper semicontinuity of P at σ. Thus, P_σ is compact.

(ii) Choose a parameter σ_0 in $\tilde{\mathfrak{L}}$. By Proposition 2.3, there exists an open neighborhood $W_0' \subset \mathfrak{L}$ of σ_0 such that for each $\sigma \in W_0'$ the parameter σ satisfies the Slater condition.

Let W be a compact neighborhood of P_{σ_0}, which is contained in $U_{\sigma_0} \times \mathbb{R}$. Define the real number

$$\alpha := \min\{\langle C_0(s), v\rangle \in \mathbb{R} \mid s \in S \text{ and } v \in W\} > 0.$$

By part (i) of this proposition, the mapping

$$P: \mathfrak{L} \to \mathrm{POW}(S^{N-1} \times \mathbb{R})$$

is upper semicontinuous at σ_0. Hence, there exists a neighborhood $W_0'' \subset \mathfrak{L}$ of σ_0 such that

$$\forall_{\sigma \in W_0''} P_\sigma \subset W.$$

We can assume that W_0'' is contained in the open set

$$W_0' \cap \left\{\sigma \in \mathfrak{L} \,\middle|\, \|\sigma - \sigma_0\| < \frac{\alpha}{2}\right\},$$

which implies that each $\sigma \in W_0''$ also satisfies the Slater condition.

We claim that each $P_\sigma, \sigma \in W_0''$, is closed. In fact, let (v_n, E_σ) be a sequence in P_σ such that

$$(v_n, E_\sigma) \to (v_0, E_\sigma).$$

By compactness of W, the element (v_0, E_σ) is contained in W. Thus, the element v_0 satisfies for each $s \in S$ the inequality

$$\begin{aligned}\langle C(s), v_0\rangle &= \langle C_0(s), v_0\rangle - \langle C_0(s) - C(s), v_0\rangle \\ &\geqslant \alpha - \|C_0 - C\| > \frac{\alpha}{2} > 0,\end{aligned}$$

which implies $v_0 \in U_\sigma$. By Proposition 5.1, (v_0, E_σ) is contained in P_σ. Thus, P_σ is compact and the neighborhood W_0'' of σ_0 is contained in $\tilde{\mathfrak{L}}$, i.e., $\tilde{\mathfrak{L}}$ is open. ∎

Remark. The assumption $\# S \geqslant N-1$ was only used in part (i), (2) of the proof. Further we remark, that in part (i), (2) of the proof, we used in Case 1 only variations of x in the set

$$\{r + \lambda(x - r) \in C(S) \mid \lambda \geqslant 1\},$$

and in Case 2 only variations of x in the set

$$\{r + \lambda(x_1 - r) \in C(S) \mid \lambda \geqslant 0\},$$

since the variations considered in Case 2 can be written as

$$x_n = r - \frac{\Theta}{n} = r + \frac{1}{n}(x_1 - r)$$

with $x_1 = r - \Theta$ resp.

$$x_n = r - \frac{\delta\Theta}{n} = r + \frac{1}{n}(x_1 - r)$$

with $x_1 = r - \delta\Theta$.

Thus, if the Slater condition is fulfilled then part (i), (2) of the proof works also with the weaker assumption of upper semicontinuity of P restricted to the set

$$\tilde{\mathfrak{L}}_{B, C, \gamma} := \{(B, C, \gamma, x) \in \mathfrak{L}\}$$

or even with the assumption of outer radial upper semicontinuity (ORU-continuity) introduced by B. Brosowski and F. Deutsch [3]. Thus, we have also

PROPOSITION 5.4. *Let $\sigma \in \mathfrak{L}$ satisfy the Slater condition. If the mapping*

$$P: \tilde{\mathfrak{L}}_{B, C, \gamma} \to \mathrm{POW}(S^{N-1} \times \mathbb{R})$$

is upper semicontinuous (or ORU-continuous) at σ, then P_σ is compact.

PROPOSITION 5.5. *If $\sigma \in \mathfrak{L}$ and P_σ is compact, then σ is normal.*

Proof. Let $(v, E_\sigma) \in P_\sigma$ and, by way of contradiction, suppose dim $H_v \geq 2$. Then there exists an element $w \in H_v$ such that w and v are linearly independent.

Since for $\varepsilon > 0$ small enough we have

$$\forall_{s \in S} \langle C(s), v + \varepsilon w \rangle > 0,$$

we can assume

$$\forall_{s \in S} \langle C(s), w \rangle > 0.$$

Let $s_0 \in S$ and $\lambda_0 \in \mathbb{R}$ be given by

$$\lambda_0 := \frac{\langle C(s_0), v \rangle}{\langle C(s_0), w \rangle} := \min_{s \in S} \frac{\langle C(s), v \rangle}{\langle C(s), w \rangle} > 0,$$

and consider a sequence (λ_n) such that

$$0 < \lambda_n < \lambda_0 \qquad \text{and} \qquad \lambda_n \to \lambda_0.$$

For each $n \in \mathbb{N}$ we have

$$\langle r(s)\, C(s) - B(s), v - \lambda_n w \rangle = 0,$$

where $r := \langle B, v \rangle / \langle C, v \rangle$.

Since

$$\underset{s \in S}{\forall} \frac{\langle C(s), v \rangle}{\langle C(s), w \rangle} \geqslant \lambda_0 > \lambda_n$$

implies

$$\underset{s \in S}{\forall} \langle C(s), v - \lambda_n w \rangle > 0.$$

we have

$$r = \frac{\langle B, v - \lambda_n w \rangle}{\langle C, v - \lambda_n w \rangle}.$$

This implies $(w_n, E_\sigma) \in P_\sigma$ for $w_n := (v - \lambda_n w)/\| v - \lambda_n w \|$.

Since P_σ is compact,

$$\left(\frac{v - \lambda_0 w}{\| v - \lambda_0 w \|}, E_\sigma \right) \in P_\sigma.$$

This contradicts

$$\langle C(s_0), v - \lambda_0 w \rangle = 0. \quad \blacksquare$$

COROLLARY 5.6. *If P is upper semicontinuous at $\sigma \in \mathfrak{L}$, then U_σ contains normal elements.*

Proof. This is an immediate consequence of Propositions 5.3 and 5.5. $\blacksquare$

COROLLARY 5.7. *Define the set*

$$\mathfrak{L}^* := \{ \sigma \in \mathfrak{L} \mid \# P_\sigma = 1 \textit{ and } \sigma \textit{ satisfies Slater condition} \}.$$

Then σ is normal and the mapping

$$P\colon \mathfrak{L}^* \to S^{N-1} \times \mathbb{R}$$

is continuous.

6. Continuity Properties of Q

PROPOSITION 6.1. *The mapping $R_\sigma: U_\sigma \to C(S)$ restricted to the normal points of U_σ is an homeomorphism.*

Proof. Let

$$\hat{U} := \{v \in U_\sigma \mid v \text{ is normal}\}$$

and denote by $\hat{R}_\sigma$ the restriction of R_σ to $\hat{U}$. It is clear that $\hat{R}_\sigma$ is continuous and injective.

To prove that it is homeomorphism, it suffices to prove that it is also an open mapping. In fact, let $W \subset \hat{U}$ be an open subset. Suppose by way of contradiction that $\hat{R}_\sigma(W)$ is not open in $\hat{R}_\sigma(\hat{U})$. Then there exist an element $r_\sigma := \hat{R}_\sigma(v_0)$ in $\hat{R}_\sigma(W)$ and a sequence (r_n) with $r_n \notin \hat{R}_\sigma(W)$ and $r_n \to r_0$. Let $v_n \in \hat{U}$ be such that $r_n = \hat{R}_\sigma(v_n)$. Since the sequence (v_n) is bounded, we can assume $v_n \to \bar{v}$.

Case 1:

$$\forall_{s \in S} \langle C(s), \bar{v} \rangle > 0.$$

In this case $\bar{v} \in U_\sigma$ and, by continuity,

$$r_n \to \frac{\langle B, \bar{v} \rangle}{\langle C, \bar{v} \rangle},$$

which implies $r_0 = \langle B, \bar{v} \rangle / \langle C, \bar{v} \rangle$. Since r_0 is a normal point, we have $\bar{v} = v_0$. Since W is open and $v_0 \in W$, for n large enough, $v_n \in W$, which implies $r_n \in \hat{R}_\sigma(W)$, contradicting $r_n \notin \hat{R}_\sigma(W)$.

Case 2:

$$\exists_{s_0 \in S} \langle C(s_0), \bar{v} \rangle = 0.$$

In this case $\bar{v} \notin U_\sigma$. For each $n \in \mathbb{N}$, we have

$$\forall_{s \in S} \langle r_n(s)\, C(s) - B(s), v_n \rangle = 0,$$

which implies

$$\forall_{s \in S} \langle r_0(s)\, C(s) - B(s), \bar{v} \rangle = 0.$$

This means $\bar{v} \in H_{v_0}$. Since $\dim(H_{v_0}) = 1$, $v_0 \in H_{v_0}$, and $\|\bar{v}\| = 1$, we have $\bar{v} = \lambda v_0$ for some $\lambda \neq 0$. This implies

$$\langle C(s_0), \lambda v_0 \rangle = 0$$

and hence

$$\langle C(s_0), v_0 \rangle = 0,$$

which is not possible. ▮

Remark. In the special case of Chebyshev-approximation by generalized functions (compare Example 1.1.) this result is due to E. W. Cheney and H. L. Loeb [12].

Remark. The mapping R_σ is in general neither closed nor open as the following example shows.

Choose $S = [0, 1]$, $N = 3$, and define

$$\underset{s \in S}{\forall} B(s) := (1, 0, 0) \qquad \text{and} \qquad C(s) := (0, 1, s).$$

For each $n \in \mathbb{N}$, the element

$$v_n := \left(\frac{1}{n^2}, \frac{1}{n}, 1\right) \Big/ \left\| \left(\frac{1}{n^2}, \frac{1}{n}, 1\right) \right\|$$

is contained in U_C since

$$\underset{s \in S}{\forall} \frac{1}{n} + s > 0.$$

The set $\{v_n \in U_C \mid n \in \mathbb{N}\}$ is closed (in U_C), since it has no accumulation point in U_C. The set of elements

$$r_n(s) := \frac{\langle B(s), v_n \rangle}{\langle C(s), v_n \rangle} = \frac{1/n^2}{1/n + s}$$

is not closed in $C(S)$, since it has the function $r_0(s) \equiv 0$ as a limit point.

Consider the non-normal element

$$w := \left(0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right).$$

Choose $\varepsilon = 1/10$ and define the open neighborhood W of w by setting

$$W := \{v \in U_C \mid \|v - w\| < \varepsilon\}.$$

Then $R_\sigma(W)$ is not open. In fact, if we consider

$$v_n := \frac{(1/n^2, 1/n, 1)}{\|(1/n^2, 1/n, 1)\|},$$

for n large enough, we have $r_n := R_\sigma(w_n)$ is not contained in $R_\sigma(W)$. But we have also $r_n \to 0$ and $R_\sigma(w) = 0$ is contained in $R_\sigma(W)$. Consequently, $R_\sigma(W)$ is not open.

COROLLARY 6.2. *Let σ be in $\mathfrak{L}$. Then P_σ is compact if and only if Q_σ is compact and σ is normal.*

Proof. The result is an immediate consequence of Propositions 6.1 and 5.5. ▮

PROPOSITION 6.3. (i) *If the mapping $Q: \mathfrak{L} \to \mathrm{POW}(C(S) \times \mathbb{R})$ is upper semicontinuous at $\sigma_0 \in \mathfrak{L}$, then σ_0 satisfies the Slater condition and $Q_{\sigma 0}$ is compact.*

(ii) *If $\sigma_0 \in \mathfrak{L}$ satisfies the Slater condition, $Q_{\sigma 0}$ is compact, and σ_0 is normal, then Q is upper semicontinuous at σ_0.*

(iii) *The set*

$$\tilde{\mathfrak{L}} := \{\sigma \in \mathfrak{L} \mid \sigma \text{ satisfies the Slater condition and } Q_\sigma \text{ compact and } \sigma \text{ normal}\}$$

is open in $\mathfrak{L}$.

Proof. (i) Using the remark after Proposition 4.2, we have also that σ_0 satisfies Slater condition. Suppose $Q_{\sigma 0}$ is not compact. Then there exists a sequence of points $(r_n, E_{\sigma 0})$ in Q_σ without a limit point in $Q_{\sigma 0}$. For $n \in \mathbb{N}$, define

$$\lambda_n := 1 + \frac{1}{n}, \qquad x_n := r_n + \lambda_n(x - r_n), \qquad \sigma_n := (B, C, \gamma, x_n).$$

By Lemma 5.2, $(v_n, \lambda_n E_{\sigma 0}) \in Q_{\sigma_n}$. Since $Q_{\sigma 0}$ is not compact, we have $x \notin V_{\sigma 0}$ and, consequently, $E_{\sigma 0} > 0$. Thus, we have $(r_n, \lambda_n E_{\sigma 0}) \notin Q_{\sigma 0}$. Define the open set

$$W := C(S) \times \mathbb{R} \setminus \{(r_n, \lambda_n E_{\sigma 0})\}.$$

Then we have $Q_{\sigma 0} \subset W$ and $Q_{\sigma_n} \not\subset W$ for each $n \in \mathbb{N}$. Since

$$\begin{aligned} \|\sigma_n - \sigma_0\| &= \|x_n - x\|_\infty \\ &= (\lambda_n - 1)\|r_n - x\|_\infty \\ &\leqslant \|\gamma\|_\infty E_{\sigma 0}(\lambda_n - 1) \\ &= \frac{1}{n}\|\gamma\|_\infty E_{\sigma 0}, \end{aligned}$$

it follows that $\sigma_n \to \sigma_0$, which contradicts the upper semicontinuity of P at σ_0.

(ii) Assume Q is not upper semicontinuous at σ_0. Then there exist an open set $W \subset C(S) \times \mathbb{R}$, a sequence (σ_n) in $\mathfrak{L}$, and a sequence (r_n) such that

$$W \supset Q_{\sigma_0} \quad \text{and} \quad \sigma_n \to \sigma_0 \quad \text{and} \quad r_n \in Q_{\sigma_n} \setminus W.$$

Let $v_n \in U_{\sigma_n}$ be such that

$$r_n = \frac{\langle B_n, v_n \rangle}{\langle C_n, v_n \rangle}.$$

By Corollary 6.2, P_{σ_0} is compact and, consequently, by Proposition 5.3, P is upper semicontinuous at σ_0. Choose a compact neighborhood W_1 of P_{σ_0}, which is contained in $U_{\sigma_0} \times \mathbb{R}$. By Proposition 5.3, there exists a neighborhood $W_2 \in \mathfrak{L}$ of σ_0 such that for each $\sigma \in W_2$, P_σ is compact and, by upper semicontinuity of P at σ_0, is contained in W_1. Since $\sigma_n \to \sigma_0$, for n large enough, $(v_n, E_{\sigma_n}) \in W_1$. By compactness of W_1, we can assume

$$(v_n, E_{\sigma_n}) \to (v_0, \hat{E}).$$

Since $W_1 \subset U_{\sigma_0} \times \mathbb{R}$, we have

$$\forall_{s \in S} \ \langle C_0(s), v_0 \rangle > 0$$

and, by Proposition 5.1, $(v_0, \hat{E})$ in P_{σ_0}. Then $v_n \to v_0$ implies that the sequence

$$r_n = \frac{\langle B_n, v_n \rangle}{\langle C_n, v_n \rangle}$$

converges to $\langle B_0, v_0 \rangle / \langle C_0, v_0 \rangle$, which is contained in Q_{σ_0}.

But this is impossible, since each r_n is not contained in the open set W and $Q_{\sigma_0} \subset W$. Thus, Q is upper semicontinuous at σ_0.

(iii) Choose an element σ_0 in $\tilde{\mathfrak{L}}$. By Corollary 6.2, P_{σ_0} is compact. Then, by Proposition 5.3 there exists an open set W such that $\sigma_0 \in W$ and for each $\sigma \in W$ the parameter σ satisfies the Slater condition and P_σ is compact. By Corollary 6.2, Q_σ is compact and σ normal, i.e., $W \subset \tilde{\mathfrak{L}}$. Thus, $\tilde{\mathfrak{L}}$ is open. ▌

Remark. As in the proof of part (i), (2) of Proposition 5.3 we used in part (i), only variations of x in the set

$$\{r + \lambda(x - r) \in C(S) \mid \lambda \geq 1\}.$$

Thus, if the Slater condition is fulfilled then part (i) of the proof works also with the weaker assumption of upper semicontinuity of Q restricted to the set

$$\tilde{\mathfrak{L}}_{B,C,\gamma} := \{(B, C, \gamma, x) \in \mathfrak{L}\}$$

or even with assumption of outer radial upper semicontinuity (ORU-continuity) introduced in [3]. Then, we have also

PROPOSITION 6.4. *Let $\sigma \in \mathfrak{L}$ satisfy the Slater condition. If the mapping*

$$Q : \tilde{\mathfrak{L}}_{B,C,\gamma} \to \mathrm{POW}(C(S) \times \mathbb{R})$$

is upper semicontinuous (or ORU-continuous) at σ, then Q_σ is compact.

COROLLARY 6.5. *Define the set*

$$\mathfrak{L}^{\#} := \{\sigma \in \mathfrak{L} \mid \# Q_\sigma = 1 \text{ and } \sigma \text{ satisfies the Slater condition and } \sigma \text{ normal}\}.$$

Then the mapping

$$Q : \mathfrak{L}^{\#} \to C(S) \times \mathbb{R}$$

is continuous.

PROPOSITION 6.6. (i) *If σ satisfies the Slater condition, $\pi_1 \circ Q_\sigma$ is compact, and σ is normal, then $\pi_1 \circ Q_\sigma$ is upper semicontinuous at σ.*

(ii) *The set*

$$\tilde{\mathfrak{L}} := \{\sigma \in \mathfrak{L} \mid \sigma \text{ satisfies the Slater condition and } \pi_1 \circ Q_\sigma \text{ compact and } \sigma \text{ normal}\}$$

is open in $\mathfrak{L}$.

(iii) *Define the set*

$$\mathfrak{L}^{\#} := \{\sigma \in \tilde{\mathfrak{L}} \mid \# \pi_1 \circ Q_\sigma = 1\}.$$

Then the mapping

$$Q : \mathfrak{L}^{\#} \to C(S)$$

is continuous.

Proof. The proof follows from Proposition 6.3, since Q_σ is compact if and only if $\pi_1 \circ Q_\sigma$ is compact. ∎

COROLLARY 6.7. *In the case of ordinary rational Chebyshev approximation, we have*

If σ is normal and $\# \pi_1 \circ Q_\sigma = 1$, then the metric projection is continuous at σ.

Proof. The result follows from 6.6(iii), since in the case of ordinary Chebyshev approximation we have $\gamma = 1$ (compare Example 1.1), which implies the Slater condition. ▮

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